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Exterior ballistics

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EXTERIOR BALLISTICS

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EXTERIOR BALLISTICS

BY

PHILIP R. ALGER

Professor of Mathematics, U. S. Navy

SECOND THOUSAND

1906

The Lord Baltimore Press

THE FRIEDENWALD COMPANY

BALTIMORE, MD., U. S. A.

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Professor of Mathematics, U. S. N.

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PREFACE

This volume was written primarily for use as a text-book for the midshipmen at the U. S. Naval Academy, and its scope is correspondingly limited. It is hoped, however, that it contains information which will be useful to officers of the navy generally.

High angle fire is not treated of, for the reason that it is not likely to occur in naval practice, and Siacci's method, which is specially adapted to the problems of direct fire, is the only one presented for the complete solution of trajectories.

The values of the space, time, altitude and inclination integrals, forming Table I of the appended ballistic tables, were computed by Major James M. Ingalls, U. S. Army, and with his kind permission are reprinted from his "Ballistic Tables," "Artillery Circular M, series of 1893." They are based upon the well known expressions for the resistance of the air originally deduced by General Mayevski from the results of Krupp's experiments at Meppen, and afterwards extended to higher velocities by Colonel Zaboudski. Up to 1800 f. s. velocity the tabular values are practically identical with those given in Ingersoll's Ballistic Tables, which have hitherto been used at the Naval Academy and in the navy, but for higher velocities they are considerably different and are found to be more nearly in agreement with practice.

Illustrative diagrams have been used to elucidate the text whenever it was thought that they would prove helpful, and, besides numerous examples solved in the text, over 200 examples are offered for solution, their answers being given. Many of the examples are based upon actual firings, and in no case has the result of any computation been brought into agreement with experiment by modifying the value of any ballistic constant.

U. S. Naval Academy,
Department of Mechanics,
April, 1904.

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EXTERIOR BALLISTICS

CHAPTER I.

DEFINITIONS.—THE TRAJECTORY IN A NON-RESISTING MEDIUM.

1. *Ballistics* is the science of the motion of projectiles, and *Exterior Ballistics* is that branch of it which treats of their motion after they have left the gun.

The *trajectory* is the path of the projectile from the gun to the first point of impact, which is called the *point of fall.* Unless otherwise stated, the point of fall is supposed to be on the horizontal plane of the gun.

The *range* is the distance in a straight line from the gun to the point of fall; it will be denoted by R. When the point of fall is at the same level as the gun, the range is called the *horizontal range,* and will be denoted by X. Unless otherwise stated, the term range always means horizontal range.

The *line of departure* is the line in which the projectile is moving when it leaves the muzzle of the gun. It is the tangent to the trajectory at its origin, and, with modern guns, practically coincides with the axis of the bore of the gun.

The *angle of departure* is the angle between the line of departure and the horizontal plane; it will be denoted by α.

The *angle of fall* is the angle between a tangent to the trajectory at the point of fall and the horizontal plane; it will be denoted by ω.

The *angle of position* is the angle between the horizontal plane and a line from gun to target; it will be denoted by p. It is quite common to call this angle the angle of sight for the reason that the line from gun to target, which we will call the *line of position,* coincides with the line of sight of a gun which is pointed at the target by means of sights.

The *angle of elevation* is the angle between the axis of the bore

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of the gun and the line of position at the instant of firing; it will be denoted by θ.

The *angle of projection* is the angle between the axis of the bore of the gun and the line of position at the instant the projectile leaves the muzzle; it will be denoted by θ'.

The *angle of jump* is the difference between the angle of projection (θ') and the angle of elevation (θ). In other words, it is the vertical angle which the axis of the gun describes under the shock of firing during the interval from the ignition of the powder charge to the exit of the projectile; it will be denoted by j, and is positive when it increases the elevation of the gun.

The *muzzle velocity*, or *initial velocity*, is the velocity with which the projectile is supposed to leave the gun; it is measured either in meters per second (m. s.) or in feet per second (f. s.), and is often written in abbreviated form "I. V." It will be denoted by V.

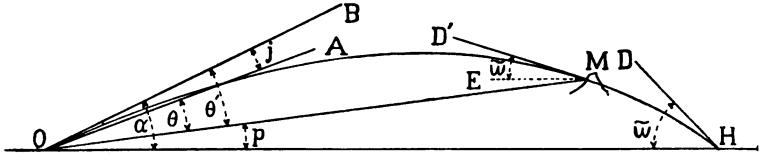

Fig. 1.

The *remaining velocity* is the velocity of the projectile at any given point in its trajectory; it will be denoted by v.

The *striking velocity* is the remaining velocity at the point of fall; it will be denoted by v_ω.

The *horizontal velocity* is the horizontal component of the velocity of the projectile at any given point in the trajectory; it will be denoted by u.

2. The two following assumptions, which are sufficiently correct for all practical purposes, are made throughout.

(a) The force of gravitation throughout the trajectory acts in parallel lines perpendicular to the horizontal plane at the gun. The value of g is taken to be 32.2 f. s. s.

(b) The dimensions of the gun are negligible in comparison with the trajectory. For convenience we will suppose the trajectory to begin at the axis about which the gun turns in being elevated or depressed, and we will take the horizontal plane through that axis to be the horizontal plane of the gun.

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3. Figure 1 represents a trajectory, O indicating the position of the gun and OH the horizontal plane. Then $BOH = \alpha$ is the angle of departure, H is the point of fall on the horizontal plane, $DHO = \omega$ is the angle of fall, and $OH = X$ is the horizontal range. If the target is at M, OM is the line of position, $MOH = p$ is the angle of position, $D'ME = \omega$ is the angle of fall, and $OM = R$ is the range. $BOM = \theta'$ is the angle of projection, which coincides with the angle of departure (α) when the angle of position (p) is zero. OA representing the position of the axis of the gun at the instant of firing, and OB its position when the projectile leaves the muzzle, the angle between them, $BOA = j$ is the angle of jump, and $AOM = \theta$ is the angle of elevation. It will be seen that $\alpha = \theta + j + p = \theta' + p$.

4. The resistance of the air to the motion of projectiles animated with the high velocities given by modern guns is so great that calculations which neglect it are seldom of any practical value. Nevertheless a study of the simple case of motion in a non-resisting medium, or vacuum, will help to a correct understanding of the more complex problems which arise when atmospheric resistance is taken account of.

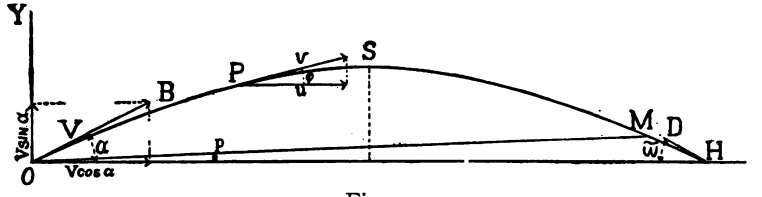

Fig. 2.

5. Figure 2 represents a trajectory in vacuum, the origin being taken at the gun, the axis of Y vertical and the axis of X horizontal. OB is the line of departure, and by its length represents the initial velocity V, the vertical and horizontal components of which are respectively V sin α and V cos α. The remaining velocity (v) at any point of the trajectory P, and its horizontal component (u), are also represented.

Since the only force acting on the projectile after it leaves the gun is the vertical force of gravitation, the projectile will remain in the vertical plane through the line of departure OB, and the trajectory will be a plane curve.

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Let t be the elapsed time from the origin to any point, whose coordinates are x and y: then evidently we have

$$\left.\begin{aligned} x &= Vt\cos\alpha \\ y &= Vt\sin\alpha - \tfrac{1}{2}gt^2 \end{aligned}\right\} \qquad (1)$$

Eliminating t between these two equations, we have

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \qquad (2)$$

And (2) is the equation of the trajectory in vacuum, which is a parabola with vertical axis.

6. Differentiating (2), and putting $\tan\phi$ for $\frac{dy}{dx}$, we have

$$\tan\phi = \tan\alpha - \frac{gx}{V^2\cos^2\alpha} \qquad (3)$$

which gives the inclination of the curve at any point.

7. Putting $y = 0$ in (2) we find two values of x, the first zero and the second $\frac{V^2\sin 2\alpha}{g}$. Consequently the horizontal range OH is given by

$$X = \frac{V^2\sin 2\alpha}{g} \qquad (4)$$

This shows that for a given initial velocity the range increases with the angle of departure up to $\alpha = 45°$, when it has its maximum value $\frac{V^2}{g}$; and that the same range is given by either of two angles of departure, one as much greater than 45° as the other is less than 45°.

8. From (3) we see that as x increases ϕ diminishes from its initial value α until for $x = \frac{X}{2} = \frac{V^2\sin 2\alpha}{2g}$ it is zero, after which it becomes negative, and for $x = X$ has the value $-\alpha$. That is to say, the highest point, or vertex (s), of the trajectory is midway of the range, and the angle of fall (ω) equals the angle of departure (α).

9. The maximum ordinate is obtained by putting $x = \frac{X}{2} = \frac{V^2\sin 2\alpha}{2g}$ in (2), whence we have for the coordinates $(x_0 y_0)$ of the vertex

$$\left.\begin{aligned} x_0 &= \frac{V^2\sin 2\alpha}{2g} = \frac{X}{2} \\ y_0 &= \frac{V^2\sin^2\alpha}{2g} = \frac{X\tan\alpha}{4} \end{aligned}\right\} \qquad (5)$$

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This value of y_0 is directly given from the fact that the vertical component of the initial velocity is $V \sin \alpha$, since the height to which a body will rise in vacuum is the square of its vertical velocity divided by $2g$.

10. The vertical and horizontal components of the velocity at any time being respectively $\frac{dy}{dt} = V \sin \alpha - gt$ and $\frac{dx}{dt} = V \cos \alpha$, we have for the remaining velocity $v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = V^2 - 2gt\,V \sin \alpha + g^2t^2 = V^2 - 2g\left(Vt \sin \alpha - \frac{gt^2}{2}\right)$, whence

$$v = \sqrt{V^2 - 2gy} \qquad (6)$$

Thus we see that where $y = 0$, both at the origin and at the point of fall on the horizontal plane, $v = V$, or the striking velocity (v_ω) is the same as the initial velocity (V). Also, putting $y = y_0 = \frac{V^2 \sin^2 \alpha}{2g}$, we have for the velocity at the vertex $v_0 = V \cos \alpha$, which, of course, it must be, since at that point the vertical component of the velocity has been destroyed by gravitation while the original horizontal component of the velocity remains unchanged throughout the trajectory.

11. Since the horizontal velocity is constant, it is evident that the time to any point whose abscissa is x is given by $t = \frac{x}{V \cos \alpha}$, and so the duration of the trajectory, or time of flight (T), is given by

$$T = \frac{X}{V \cos \alpha} = \sqrt{\frac{2X \tan \alpha}{g}} \qquad (7)$$

The latter of these values is obtained directly from the consideration that if gravity did not act the projectile would rise to the height $X \tan \alpha$ while moving X horizontally, and so the distance which it falls from the tangent under the action of gravity during the time of flight (T) is given by $X \tan \alpha = \frac{1}{2}gT^2$.

12. To determine the range (R) on an inclined plane let x_1 y_1 be the coordinates of the target M (Figure 2), p being the angle of position and $\theta' = \alpha - p$ the angle of projection. Then from (2) we get

$$\tan p = \frac{y_1}{x_1} = \tan \alpha - \frac{gx_1}{2V^2 \cos^2 \alpha}$$

$$x_1 = \frac{2V^2}{g} \cos^2 \alpha \,(\tan \alpha - \tan p).$$

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But $\tan \alpha - \tan p = \frac{\sin \alpha \cos p - \cos \alpha \sin p}{\cos \alpha \cos p} = \frac{\sin (\alpha - p)}{\cos \alpha \cos p}$, therefore

$$x_1 = \frac{2V^2}{g} \cdot \frac{\sin (\alpha - p) \cos \alpha}{\cos p} = \frac{2V^2}{g} \cdot \frac{\sin \theta' \cos (\theta' + p)}{\cos p}$$

whence, since $R = x_1 \sec p$, we have

$$R = \frac{2V^2}{g} \cdot \frac{\sin \theta' \cos (\theta' + p)}{\cos^2 p} \qquad (8)$$

13. If p were zero, so that the angle of departure was θ', we should have for the range $X_1 = \frac{2V^2}{g} \cdot \sin \theta' \cos \theta'$, and so the ratio of the range on an inclined plane to the horizontal range, *for the same angle of projection,* is given by

$$\frac{R}{X_1} = \frac{\cos (\theta' + p)}{\cos \theta' \cos^2 p} = \sec p \, (1 - \tan \theta' \tan p) \qquad (9)$$

The value of the second member of (9) is very nearly unity so long as θ' and p are small angles, being, for example, .9992 for $\theta' = 3°$, $p = 5°$, and it follows that with small angles of projection and position, we may consider the range to be independent of the value of the angle of position.

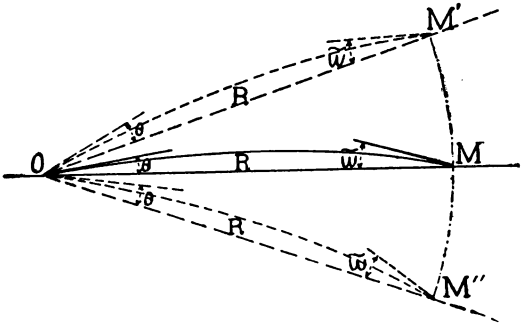

Fig. 3.

This is called the assumption of the *rigidity of the trajectory,* and consists, evidently, in supposing that the gun, trajectory, and line of position, a chord of the trajectory, may be turned through a vertical angle, as illustrated in Figure 3, without any change of form.

The trajectory is assumed to be rigid in practice when the same sight graduations are used in firing a gun at objects at different heights, or with the gun itself at different heights.

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EXAMPLES.—I.

(1) If the angle of elevation is 3° and the jump 5′, what is the angle of projection, and if the position angle is 2° 25′, what is the angle of departure?

$\theta' = 3°\ 05'$; $\alpha = 5°\ 30'$.

(2) If the I. V. is 2000 f. s., and the angle of departure is 5°, what are the horizontal and vertical components of V ?

$V_x = 1992.4$ f. s.; $V_y = 174.3$ f. s.

(3) The angle of departure is 7° 20′, the position angle is 4° 10′, and the jump is 3′, what are the angles of projection and elevation? $\theta' = 3°\ 10'$; $\theta = 3°\ 07'$.

(4) A body is projected in vacuum with V = 1000 f. s. and $\alpha = 30°$, where is it after 3 seconds? After 10 seconds?

$x = 1500\sqrt{3}$ ft., $y = 1355$ ft.
$x = 5000\sqrt{3}$ ft.; $y = 3390$ ft.

(5) What is the height of vertex and what the range in the case of example 4? $y_0 = 3882$ ft.; $X = 15528\sqrt{3}$ ft.

(6) A body is projected in vacuum from the top of a tower 200 ft. high with a velocity of 50 f. s. and an angle of departure of 60°. Find the range on the horizontal plane through the foot of the tower, and the time of flight. 128 ft., 5.12 sec.

(7) The horizontal range in vacuum is 1000 ft. and the time of flight is 15 sec. What were the angle of departure, the I. V., and the height of vertex?

$\alpha = 74°\ 34'$; $V = 250.5$ f. s.
$y_0 = 905.6$ ft.

(8) What must the angle of departure be that the horizontal range in vacuum may equal three times the greatest altitude? What that the range may equal the altitude?

$\alpha = \tan^{-1}\frac{4}{3} = 53°\ 07'\ 48''$
$\alpha = \tan^{-1} 4 = 75°\ 57'\ 50''$.

(9) Find the I. V. and the angle of departure in vacuum that the projectile may be 100 ft. high at the horizontal distance of a quarter of a mile and may have a horizontal range of one mile.

$V = 922.6$ f. s.; $\alpha = 5°\ 46'$.

(10) The angle of departure is 1° 16′ 31″ and the I. V. is 1500 f. s.; find the range and time of flight, in vacuum.

$X = 1036$ yds.; $T = 2.074$ sec.

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(11) The angle of position is 45°, the angle of projection 1° 16′ 31″, and the I. V. is 1500 f. s.; find the range on the incline and the time of flight, in vacuum.

R = 1433 yds.; T = 2.93 sec.

(12) What must be the angle of projection in vacuum for an I. V. of 400 f. s. that the range on a plane which descends at an angle of 30° may be 2500 yds.?

4° 36′ or 55° 24′.

(13) A body is projected in vacuum with $\alpha = 60°$ and V = 150 f. s.; find the coordinates of its position, the direction of its motion, and its velocity after 5 seconds.

x = 375 ft.; y = 247.4 ft.
$\phi = -22° 31'$; v = 81.2 f. s.

(14) A 12-inch mortar shell weighing 610 lbs. fired with an I. V. of 591 f. s. and an angle of departure of 73°, gave an observed horizontal range of 1939 yds. and a time of flight of 36 sec. What would the range and time of flight have been in vacuum?

X = 2022 yds.; T = 35.1 sec.

(15) The measured range of a 12-inch shell of 850 lbs. weight fired with 2800 f. s. I. V. and an angle of departure of 7° 32′, was 11,900 yds., and its time of flight 19.5 secs. What would the range and time of flight have been in vacuum?

X = 21,097 yds.; T = 22.8 sec.

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CHAPTER II.

THE RESISTANCE OF THE AIR.

14. Historical Sketch.—The ancient artillerists supposed that the trajectory was composed of three distinct parts, of which the first, or "violent" part, was a straight line, the middle a circle, and the last, or "natural" part, again straight.

Nicholas Tartaglia, who published a treatise on the flight of projectiles in 1537, may well be called the father of ballistics. He invented the quadrant, for measuring a gun's angle of elevation, and showed that the trajectory is curved throughout and that the range increases with the angle of departure up to 45°, when it is greatest (in vacuum).

Galileo, who also disregarded the resistance of the air, proved that the path of a projectile is a parabola, with vertical axis.

Newton investigated the trajectory on the hypothesis that the resistance of the air is proportional to the velocity, and showed that, instead of being a parabola, it had two dissimilar branches, of which the descending branch, if continued indefinitely, would become a vertical straight line. He also experimented with falling bodies and concluded that the resistance of the air was proportional to the velocity squared.

John Bernoulli, in 1718, published a mathematical solution of the problem of the trajectory in air supposing the resistance to vary as the nth power of the velocity, but his results were in a form not well suited for practical use.

15. *Robins* was the first to measure experimentally, with any degree of accuracy, the velocities of projectiles. In 1742 he published his famous "New Principles of Gunnery," and described his invention, the "ballistic pendulum," which consisted of a heavy bob suspended from a tripod and arranged to receive the impact of a projectile and to register the resulting swing; the velocity imparted to the bob by the impact could then be calculated, and thence, by the principle of the equality of the quantities of motion before and after the impact, the striking velocity of the projectile could be determined. Robins measured velocities as great as 1700 f. s., and determined approximately the loss of velocity due to air resistance for distances up to 250 feet from the gun. He also suggested using the gun itself as a pendulum and calculating the velocity of the projectile by observing the

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swing of the gun when it was fired, and *Count Rumford* soon afterwards experimented with this device.

Dr. Hutton made extensive experiments in England from 1775 to 1791 both with the ballistic pendulum and with the gun pendulum, but there was no further great advance in knowledge of ballistics until the French government experiments were made at Metz in 1839 and 1840, from the results of which it became possible to construct trajectories roughly and to calculate, by empirical formulæ, approximate ranges and times of flight for any given elevation of a gun.

16. In 1840 *Wheatstone* suggested measuring the velocity of a projectile by causing it to cut successive wire screens, each of which formed part of an electric circuit, and, in following years, many instruments for measuring projectile velocities by electricity were perfected, of which the *Boulengé Chronograph* is to-day the most widely used. This instrument consists of two electro-magnets, the magnetizing currents of which pass through wire screens at a carefully measured distance apart in the path of the projectile, and from one of which is suspended a long cylindrical rod called the ***chronometer,*** while from the other is suspended a shorter rod called the ***registrar.*** A spring trigger carrying a knife edge is so arranged that when the registrar drops it strikes and releases it, causing it to fly forward and make a cut on the side of the chronometer. When released by hand the trigger makes a mark on the suspended chronometer called the *origin;* when the circuits are simultaneously broken by a device called the *disjunctor,* the chronometer and registrar fall freely side by side until the released trigger makes a second mark on the chronometer at a distance above the origin called the *disjunctor reading.* When a projectile is fired it cuts the first screen, releasing the chronometer, then it cuts the second screen, releasing the registrar, and thus a mark is made whose distance above the origin is called the *velocity reading;* then the time of falling freely the velocity reading, less the time of falling freely the disjunctor reading, is the time it took the projectile to travel from the first to the second screen, and the distance between the screens divided by this time is taken to be the velocity of the projectile when midway.

17. The perfection of a means of accurately measuring the velocity of a projectile at any desired point in its path made it practicable to experimentally investigate the laws governing the resistance of the air.

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Bashforth, in very extensive experiments (1865-1870 and 1878-1880), showed that the resistance of the air was proportional to the square of the projectile's diameter, and that for flat, hemispherical, and ogival shaped heads it varied in ratios about as 1.5, 1.0, and 0.8. His earlier experiments were unfortunately made with studded projectiles, which seem to have had insufficient rotation for smooth flight, but the ballistic tables based upon his experimental results are still used to some extent for calculating trajectories.

18. Mayevski's results, published in 1883, and founded upon Krupp's experiments made at Meppen in 1881, are considered the most reliable up to date, and, as modified and extended to higher velocities by Colonel Zaboudski, they have been adopted for use in this book.

Expressing the retardation caused by the resistance of the air in the form $A \frac{d^2}{w} v^n$, in which d is the diameter of the projectile in inches, w its weight in pounds, and v its velocity in f. s., Mayevski's equations are as follows:

$$
\left.
\begin{aligned}
&\textit{v between 3600 f. s. and 2600 f. s.}\\
&\frac{dv}{dt} = -A_1 \frac{d^2}{w} v^{1.55} \qquad \log A_1 = 7.6090480\ (-10)\\
&\textit{v between 2600 f. s. and 1800 f. s.}\\
&\frac{dv}{dt} = -A_2 \frac{d^2}{w} v^{1.7} \qquad \log A_2 = 7.0961978\ (-10)\\
&\textit{v between 1800 f. s. and 1370 f. s.}\\
&\frac{dv}{dt} = -A_3 \frac{d^2}{w} v^{2} \qquad \log A_3 = 6.1192596\ (-10)\\
&\textit{v between 1370 f. s. and 1230 f. s.}\\
&\frac{dv}{dt} = -A_4 \frac{d^2}{w} v^{3} \qquad \log A_4 = 2.9809023\ (-10)\\
&\textit{v between 1230 f. s. and 970 f. s.}\\
&\frac{dv}{dt} = -A_5 \frac{d^2}{w} v^{5} \qquad \log A_5 = 6.8018712\ (-20)\\
&\textit{v between 970 f. s. and 790 f. s.}\\
&\frac{dv}{dt} = -A_6 \frac{d^2}{w} v^{3} \qquad \log A_6 = 2.7734430\ (-10)\\
&\textit{v between 790 f. s. and 0 f. s.}\\
&\frac{dv}{dt} = -A_7 \frac{d^2}{w} v^{2} \qquad \log A_7 = 5.6698914\ (-10)
\end{aligned}
\right\} \quad (10)
$$

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19. The projectiles used in the determination of the foregoing values of the coefficient A and the exponent n were about three calibers long, had ogival heads of two calibers radius, and were given rotation by forced bands, thus resembling in all respects most of the projectiles in modern use. Consequently, in order to apply Mayevski's equations to projectiles differing in form from the standard just described, it is necessary to introduce into them a *coefficient of form* (i).

20. Furthermore, the values of A and n correspond to a standard density of the air, and, in order to express the retardation due to other than this standard atmospheric condition, it is necessary to introduce another coefficient (δ) which is the ratio of the actual density of the air to its assumed standard density $\left(\delta = \frac{\Delta}{\Delta_0}\right)$

21. Thus the general formula for the retardation becomes

$$\frac{dv}{dt} = -A\frac{\delta i d^2}{w}v^n \qquad (11)$$

in which A and n have the numerical values already given, i depends on the form of the projectile, being unity for most modern projectiles, and δ depends upon the atmospheric state at the time considered.

22. Since for any given projectile the retardation is inversely proportional to $\frac{w}{id^2}$, that factor, which is called the *ballistic coefficient,* and is denoted by C, measures the *ranging power* of a projectile. With spherical projectiles C could only be increased by increase of caliber, but with rifled guns it became possible to greatly increase its value for any caliber of gun by lengthening the projectile and sharpening its point, and it is this fact which largely accounts for the immense superiority in ballistic power of rifles over smooth bores.

23. Of course the resistance of the air in pounds (R) is found by multiplying the retardation by the mass of the projectile, so that we have

$$R = A\frac{\delta i d^2}{g}v^n \qquad (12)$$

For example, to find the resistance offered by the air, at standard density, to a standard 12-inch projectile, of 850 pounds weight, at 2000 f. s. velocity, we have $\delta = 1$, $i = 1$, $d = 12$, $v = 2000$, $g = 32.2$, and from (10) $\log A = 7.09620(-10)$, and

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$n = 1.7$; whence we find $R = 2282.7$ pounds; while the retardation, or loss of velocity per second in f. s., due to this resistance, is $\frac{R}{m} = \frac{Rg}{w} = 86.47$ f. s.

24. Experimental Determination of the Resistance of the Air.—To measure the resistance of the air to a given projectile the gun is given a very slight elevation, and, by means of electric chronographs, the velocities (v_1 and v_2) of the projectile are measured at two points sufficiently far apart to make the velocities appreciably different and yet near enough together for the resistance of the air not to have changed much from one to the other. The two pairs of screens used for measuring the velocities should be at about the same level so that the effect of gravitation may be neglected.

Let w be the weight of the projectile, R the resistance of the air, and x the distance between the two points where the velocities v_1 and v_2 are determined. Then, since the work done by the resistance must equal the loss of energy of the projectile, we have

$$Rx = \frac{w}{2g}(v_1^2 - v_2^2)$$

$$R = \frac{w}{2gx}(v_1^2 - v_2^2) \qquad (13)$$

and R is taken to be the resistance of the air which corresponds to the mean velocity $\frac{v_1 + v_2}{2}$.

25. As the result of many such measurements with different projectiles and different velocities, it has been shown that the resistance of the air is proportional to—

1°. The cross sectional area of the projectile, or, what is the same thing, the square of the caliber (d^2).

2°. The density of the air, or, what is the same thing, the weight of a cubic foot of the air (Δ).

3°. A power of the velocity (v^n) of which the exponent n varies with the velocity, but may be considered constant for velocities within certain limits.

4°. A coefficient (A) which varies with the velocity, with the form of the projectile, and with the assumed value of the exponent n, but which may be considered constant for any given projectile between the same limits of the velocity for which n is considered constant.

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And it is in accordance with these experimentally determined laws that we wrote equation (11), expressing the retardation caused by the resistance of the air in the form $A\frac{\delta id^2}{w}v^n$, or $A\frac{\delta}{C}v^n$, and in which A and n are to be given the numerical values set forth in (10).

26. As already stated, the product $\frac{w}{id^2}$ is known as the *ballistic coefficient* and denoted by C: with the additional factor $\frac{1}{\delta}$, which corrects for any variations of the density of the air from the assumed standard density, and with certain other factors to be hereafter explained, the product is called the *reduced ballistic coefficient* and is denoted by C'. Thus as the most general expression for the retardation we have

$$\frac{dv}{dt} = -\frac{A}{C'}v^n \qquad (14)$$

EXAMPLES.—II

(1) A 6-inch projectile, weight 70 lbs., is fired through screens and the velocities measured at two points 600 ft. apart are 1951 f. s. and 1874 f. s. What was the mean resistance of the air? 533.6 lbs.

(2) A 6-inch ogival headed projectile, weight 70 lbs., is fired through screens 450 ft. apart and its velocities at the first and second pairs of screens are respectively 1846 f. s. and 1790 f. s. A 6-inch flat-headed projectile of the same weight, fired through the same screens, gave measured velocities of 1929 f. s. and 1790 f. s. What was the resistance to each projectile?

491.8 lbs.; 1248.6 lbs.

(3) A 12-inch projectile, weight 850 lbs., gave measured velocities 1979 f. s. and 1956 f. s. at points 500 ft. apart. What was the mean resistance of the air? If the density of the air at the time of the experiment was 1.02 times the standard density, what would the resistance be in a standard atmosphere?

2389 lbs.; 2342 lbs.

(4) Determine the resistance of the air, and the retardation, to a standard 3-inch projectile, weight 13 lbs., when moving at 2800 f. s. and at 2000 f. s. 250.3 lbs.; 620.1 f. s. s.

142.7 lbs.; 353.4 f. s. s.

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(5) Determine the resistance of the air, and the retardation, in the case of a standard 4-inch projectile, weight 33 lbs., when moving at 2800 f. s., and at 2000 f. s.

445.1 lbs.; 434.3 f. s. s.
253.6 lbs.; 247.5 f. s. s.

(6) Determine the resistance of the air, and the retardation, in the case of a standard 6-inch projectile, weight 100 lbs., when moving at 2800 f. s., and at 2000 f. s.

1001.4 lbs.; 322.4 f. s. s.
570.7 lbs.; 183.8 f. s. s.

(7) Determine the resistance of the air, and the retardation, in the case of a standard 8-inch projectile, weight 250 lbs., when moving at 2800 f. s., and at 2000 f. s.

1780.2 lbs.; 229.3 f. s. s.
1014.5 lbs.; 130.7 f. s. s.

(8) Determine the resistance of the air, and the retardation, in the case of a standard 10-inch projectile, weight 500 lbs., when moving at 2800 f. s., and at 2000 f. s.

2781.6 lbs.; 179.1 f. s. s.
1585.2 lbs.; 102.1 f. s. s.

(9) Determine the resistance of the air, and the retardation, in the case of a standard 12-inch projectile, weight 850 lbs., when moving at 2800 f. s., and at 2000 f. s.

4005.5 lbs.; 151.7 f. s. s.
2282.7 lbs.; 86.5 f. s. s.

(10) Determine the resistance of the air, and the retardation, of a 12.5-inch standard projectile, of weight 802¼ lbs., moving at 1400 f. s. What would the retardation be if the weight were 1000 lbs?

1251.7 lbs.; 50.2 f. s. s.
40.3 f. s. s.

(11) What is the resistance of the air to a baseball of 3 inches diameter moving at 100 f. s., supposing the resistance to a sphere to be 1¼ times that to a standard ogival? .1634 lbs.

(12) If the resistance in the case of example (1) was proportional to $v^{1.7}$, what is the value of A?

.001259

(13) If the resistance in the case of example (3) was proportional to $v^{1.7}$, what is the value of A? ($\delta = 1.02$ at time of firing).

.001317

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CHAPTER III.

THE EQUATION TO THE TRAJECTORY IN AIR.

27. Mayevski's equations (10) show that strictly speaking A and n can only be considered constants in the general expression for the retardation caused by the resistance of the air $\left(\frac{dv}{dt} = -\frac{A}{C'}v^n\right)$ (14) over certain limited ranges of the value of the velocity (v). The value of n, however, differs but little from 2 for the velocities which obtain throughout the trajectories of most modern guns, when fired with moderate elevations, and by taking

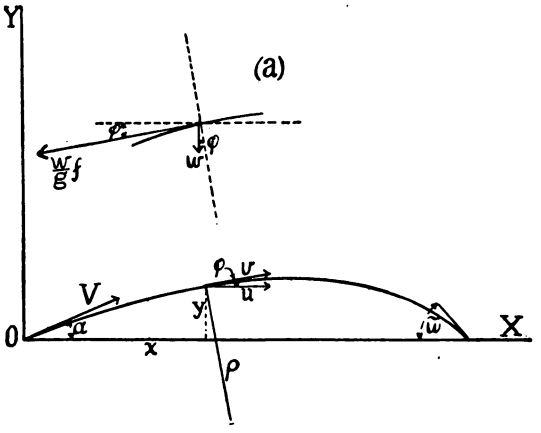

Fig. 4.

a suitable value for A the retardation may without great error be assumed to be $\frac{A}{C'}v^2$ for such trajectories. We will therefore deduce an equation to the trajectory which would result from an air resistance varying as the velocity squared, as from this equation we will be able to learn much of the form and general properties of the real trajectory in air.

28. Assuming that the axis of the projectile coincides with the tangent to its path, which is very nearly the case with modern rifled guns, the resultant action of the resistance of the air will

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likewise coincide with the axis, and the trajectory will be the same as if the mass of the projectile were concentrated at its center of gravity and moved under the action of two forces, one the constant vertical force of gravity w, and the other the variable resistance of the air $\frac{w}{g}\frac{A}{C'}v^2$ acting in the tangent. Figure 4 represents the trajectory, which of course is a plane curve under the foregoing supposition, and Figure 4 (a) represents the two forces acting on the projectile at any point, the resistance of the air being denoted by $\frac{w}{g}f$, in which f is the retardation $\left(\frac{dv}{dt}\right)$, which we are now taking to be proportional to v^2.

29. Taking vertical and horizontal axes at the point of departure O, let V be the initial velocity, α the angle of departure, v the velocity at any point whose coordinates are x, y, and u the latter's horizontal component. Then, putting $\frac{dv}{dt} = -kv^2$, so that k is the $\frac{A}{C'}$ of equation (14), since w has no horizontal component, the acceleration parallel to the X axis is given by

$$\frac{d^2x}{dt^2} = -kv^2\cos\phi \tag{15}$$

But $\frac{d^2x}{dt^2} = \frac{du}{dt}$; $v\cos\phi = u$; and $v = \frac{ds}{dt}$; whence (15) may be written

$$\frac{du}{dt} = -ku\frac{ds}{dt} \tag{16}$$

$$\frac{du}{u} = -kds \tag{17}$$

and, integrating (17) between corresponding limits of u and s, we get

$$\left.\begin{aligned} &\log_e u\Big]_u^{V\cos\alpha} = -ks\Big]_s^0 \\ &\log\frac{V\cos\alpha}{u} = ks \\ &u = V\cos\alpha\,\epsilon^{-ks} \end{aligned}\right\} \tag{18}$$

Next resolving along the normal, since the acceleration towards the center of curvature is given by the expression $\frac{v^2}{\rho}$, ρ being the radius of curvature, we have

$$\frac{v^2}{\rho} = g\cos\phi \tag{19}$$

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But $v = u \sec \phi$, and $\rho = -\frac{ds}{d\phi}$; whence (19) may be written

$$\left.\begin{aligned} u^2 \sec^2\phi d\phi &= -g \cos\phi ds = -gdx \\ \sec^2\phi d\phi &= -\frac{gdx}{u^2} \end{aligned}\right\} \quad (20)$$

Now substituting in (20) the value of u given in (18),

$$\sec^2\phi d\phi = -\frac{g}{V^2 \cos^2 a} \epsilon^{2kx} dx \quad (21)$$

30. In the case of a flat trajectory, in which the angle of departure does not exceed 4° or 5°, the difference between the values of s and x is so small that it may be practically disregarded and x may be substituted for s in (21), giving, after integration,

$$\left.\begin{aligned} \tan\phi \Big]_a^\phi &= -\frac{g}{2k\, V^2 \cos^2 a} \epsilon^{2kx} \Big]_0^x \\ \tan\phi &= \tan a - \frac{g}{2k\, V^2 \cos^2 a} (\epsilon^{2kx} - 1) \end{aligned}\right\} \quad (22)$$

But ϵ^{2kx}, when expanded by Maclaurin's theorem, equals $1 + 2kx + 2k^2x^2 + \frac{4k^3x^3}{3} + \ldots\ldots$, so that $\epsilon^{2kx} - 1 = 2kx(1 + kx + \frac{2}{3}k^2x^2 + \ldots\ldots)$, whence, substituting in (22) and writing $\frac{dy}{dx}$ for $\tan\phi$, we have

$$\frac{dy}{dx} = \tan a - \frac{gx}{V^2 \cos^2 a}(1 + kx + \tfrac{2}{3}k^2x^2 + \ldots.)$$

or, integrating between corresponding limits of x and y,

$$y = x \tan a - \frac{gx^2}{2\, V^2 \cos^2 a}(1 + \tfrac{2}{3}kx + \tfrac{1}{3}k^2x^2 + \ldots.) \quad (23)$$

But, as will be shown further on, the greatest value of kx is always a small fraction in any trajectory flat enough to justify the substitution of x for s which has already been made: hence we may neglect the terms beyond k^2x^2 in the expansion, and write for the equation of the trajectory

$$y = x \tan a - \frac{gx^2}{2\, V^2 \cos^2 a}(1 + \tfrac{2}{3}kx + \tfrac{1}{3}k^2x^2) \quad (24)$$

31. Comparing this equation with (2), it will be seen that its first two terms represent the trajectory in vacuum, and that it only differs from the latter by having other terms, subtractive like the second, and containing higher powers of x.

Figure 5 represents on the same scale the trajectories in air and in vacuum of a 13-inch projectile, weight 1100 pounds, fired

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with an initial velocity of 2300 f. s. at an angle of departure of 1° 46′ 35″, the range in vacuum being 10,196 feet and in air 9000 feet. The ordinates of both curves are exaggerated ten times as compared with the abscissæ.

If gravity did not act, the projectile would move in the tangent OQ_1Q_2, and in travelling the horizontal distance $x = OA$ would rise to the height $y = AQ_1 = x \tan \alpha$. The attraction of gravitation, however, pulls the projectile down $Q_1P_1 = \frac{gx^2}{2V^2\cos^2\alpha}$ while it moves OA horizontally, and so we have for the ordinate of the trajectory in vacuum $y_1 = AP_1 = x \tan \alpha - \frac{gx^2}{2V^2\cos^2\alpha}$. When the resistance of the air acts, retarding the motion, it takes longer for the projectile to move OA horizontally, and so it has fallen the further distance

$$P_1P' = \frac{gx^2}{2V^2\cos^2\alpha}\left(\tfrac{2}{3}kx + \tfrac{1}{3}k^2x^2\right),$$

and the ordinate of the trajectory in air is

$$y_2 = AP' = x \tan \alpha - \frac{gx^2}{2V^2\cos^2\alpha}\left(1 + \tfrac{2}{3}kx + \tfrac{1}{3}k^2x^2\right)$$

In the particular case represented, the projectile in air reaches the ground after having travelled the horizontal distance $X = OP'' = 9000$ feet; if it had moved in vacuum, it would have been at P_2, 32.23 feet above P″, after an equal horizontal travel.

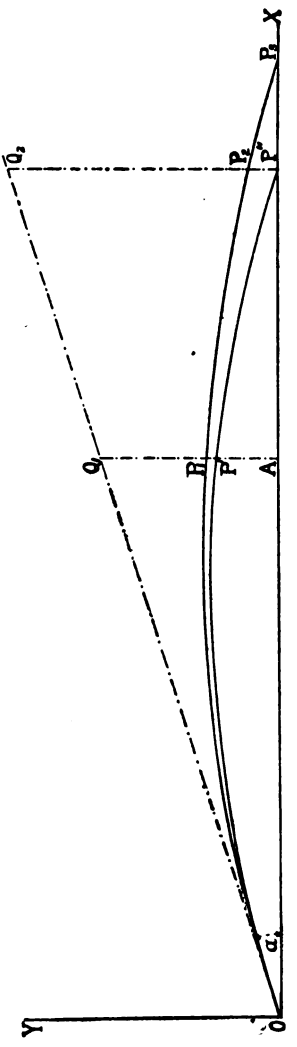

Fig. 5.

32. The value of k in the equation of the trajectory in air (24) just deduced is, as already stated, $\frac{A}{C'}$ where A is an experimen-

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tally determined constant and $C' = \frac{w}{\delta i d^2}$ is the reduced ballistic coefficient.

We will take A = .0001316, that being the value assigned by Mayevski to the constant coefficient of his expression for the retardation when the latter is proportional to v^2, as given in the third of equations (10). Then the following table gives the values of k for U. S. naval projectiles (for which i = 1) in a standard atmosphere ($\delta = 1$).

Caliber.	Weight.	$C = \frac{w}{d^2}$	$k = \frac{A}{C}$
6 pdr.	6 lb.	1.1958	.00011006
3 inch.	13 "	1.4444	.00009111
4 "	33 "	2.0625	.00006381
5 "	50 "	2.0000	.00006580
6 "	100 "	2.7778	.00004738
7 "	165 "	3.3673	00003908
8 "	250 "	3.9063	.00003369
10 "	500 "	5.0000	.00002632
12 "	850 "	5.9029	.00002229
13 "	1100 "	6.5088	.00002022

33. If we put y = 0 in equation (24), we get for values of x, one zero, denoting the origin, and another, the range X, given by

$$X(1 + \tfrac{2}{3}kX + \tfrac{1}{3}k^2X^2) = \frac{V^2 \sin 2\alpha}{g} \qquad (25)$$

But the second member of (25) is the range in vacuum for the initial velocity V and the angle of departure α, so, if we denote the range in vacuum by P, we see that the expression $1 + \tfrac{2}{3}kX + \tfrac{1}{3}k^2X^2$ equals $\frac{P}{X}$. This ratio, which will be found to play an important part in many ballistic problems, will hereafter be called n. Hence we have for the range in air the expression.

$$X = \frac{V^2 \sin 2\alpha}{gn} \qquad (26)$$

34. Since k is a very small fraction, the value of n, which is evidently unity for X = 0, increases slowly with X and for moderate values of X is only slightly greater than unity.

Thus in the case of the 3000-yard 13-inch trajectory shown in

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Figure 5 the true value of n is $\frac{P}{X} = \frac{10196}{9000} = 1.133$, while its theoretical value, using the tabular value of k given in 32, is $1 + \frac{3}{2}(.00002022 \times 9000) + \frac{1}{3}(.00002022 \times 9000)^2 = 1.132$.

35. The value of n being known, the angle of departure for any range is obtained by writing (26) in the form

$$\sin 2\alpha = \frac{gnX}{V^2} \qquad (27)$$

Thus, for example, to find the angle of departure for a range of 4000 yards, in the case of the 13-inch gun, V = 2300 f. s., w = 1100 pounds; we first calculate $n = 1 + \frac{3}{2}kX + \frac{1}{3}k^2X^2$, using the tabular value .00002022 for k and 12000 for X, thus getting n = 1.181; then $\sin 2\alpha = \frac{32.2 \times 1.181 \times 12000}{2300^2} = [\log 8.93598]$; $2\alpha = 4^\circ\ 57'$; $\alpha = 2^\circ\ 28'\ 30''$, which agrees exactly with the value given in the official range table.

36. The Rigidity of the Trajectory.—It has been shown in 13 that the range in vacuum of a projectile along any slightly inclined plane is practically the same as the horizontal range for the same angle of the projection, provided the latter angle is a small one. In precisely the same way it may be shown that this is true of the trajectory in air. Indeed, since the resistance of the air acts in the tangent to the trajectory and is independent of its inclination, it is evident that if two trajectories in vacuum are of the same form, so that one may be superposed on the other, they will still be so when each is altered by air resistance.

This principle of the rigidity of the trajectory would only be exactly true if the value of the radius of curvature was independent of the value of the angle of departure, and, as shown by (19), this is not the case, the initial value of ϕ being α and so the radius of curvature of the first part of any trajectory being greater, the greater the value of α: it is, however, sufficiently near the truth for all practical purposes so long as neither the angle of projection (θ') nor the angle of position (p) exceeds about 10°.

EXAMPLES.—III.

(1) Determine the value of the radius of curvature at the point of departure of a trajectory for which V = 2000 f. s. and $\alpha = 3^\circ$; what is its value if $\alpha = 8^\circ$?

124390 ft.
125440 ft.

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(2) In the two cases of example (1) the striking velocities are respectively 1600 f. s. and 1240 f. s., and the angles of fall are 3° 29′ and 11° 08′; what are the radii of curvature at the points of fall?

79648 ft.
48667 ft.

(3) Calculate the angles of departure for 1000, 2000, and 3000 yards range in the case of the 6-inch gun for which $k = .00004738$, and $V = 2400$ f. s.

$n = 1.101;\ 1.216;\ 1.345.$
$\alpha = 32';\ 1° 10';\ 1° 56'.$

(4) Calculate the angles of departure for 1000, 2000, and 3000 yards range in the case of the 8-inch gun for which $k = .00003369$, and $V = 2400$ f. s.

$n = 1.071;\ 1.148;\ 1.233.$
$\alpha = 31';\ 1° 06';\ 1° 47'.$

(5) Calculate the angles of departure for 1000, 2000, and 3000 yards range in the case of the 10-inch gun for which $k = .00002632$ and $V = 2400$ f. s.

$n = 1.055;\ 1.114;\ 1.177.$
$\alpha = 30';\ 1° 04';\ 1° 42'.$

(6) Calculate the angles of departure for 1000, 2000, and 3000 yards range in the case of the 12-inch gun for which $k = .00002229$ and $V = 2400$ f. s.

$n = 1.046;\ 1.095;\ 1.147.$
$\alpha = 30';\ 1° 03';\ 1° 39'.$

(7) The 6-inch gun with $V = 2900$ f. s. gave a measured range of 5394 yards for $\alpha = 3° 03' 51''$. Determine the value of n and compare it with its calculated value for the same range ($k = .00004738$).

$n = 1.723,\ 1.707.$

(8) The 6-inch gun with $V = 2900$ f. s. gave a measured range of 2625 yards for $\alpha = 1° 07' 49''$. Determine the value of n and compare it with its calculated value for the same range ($k = .00004738$).

$n = 1.308,\ 1.295.$

In the following examples find k from $n = 1 + \frac{2}{3}kX$, *neglecting higher powers of* kX.

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(9) The angle of departure of the 12-inch 850-pound projectile (V = 2800 f. s.) for a range of 5000 yards is 2° 11′. Compute the ordinates of the trajectory at 1000-yard intervals and compare them with the ordinates of the trajectory in vacuum with the same angle of departure and I. V.

In vac., 0; 95.9; 154.7; 176.5; 161.4; 109.1.
In air, 0; 95.0; 147.7; 153.0; 105.4; 0.

(10) The angle of departure of the 6-inch 100-pound projectile (V = 2900 f. s.) for a range of 3000 yards is 1° 17′. Compute the ordinates of the trajectory at 1000 yard intervals and compare them with the ordinates of the trajectory in vacuum with the same angle of departure and I. V.

In vac., 0; 50.0; 65.5; 46.5.
In air, 0; 48.2; 51.7; 0.

(11) The angle of departure of the 3-inch 13-pound projectile (V = 2800 f. s.) for a range of 2000 yards is 1° 01′. Compute the ordinates of the trajectory at 500-yard intervals and compare them with the ordinates of the trajectory in vacuum with the same angle of departure and I. V.

In vac., 0; 22.0; 34.8; 38.3; 32.5.
In air, 0; 21.5; 30.7; 24.5; 0.

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CHAPTER IV.

THE TRAJECTORY IN AIR.—APPROXIMATE DETERMINATIONS.

37. Before proceeding to the discussion of the more exact calculation of the elements of the trajectory in air by the use of ballistic tables, we will deduce formulæ by means of which those elements can be determined with a sufficient approximation for many practical purposes.

We will take as the equation of the trajectory in air that deduced in the last chapter (eq. 23) but simplified by omitting higher powers of kx than the first.

$$y = x \tan \alpha - \frac{gx^2}{2V^2 \cos^2 \alpha}(1 + \tfrac{2}{3} kx) \qquad (28)$$

In this equation k can no longer be considered strictly constant, but its value, when found for any one value of α, may be used over a considerable range of values of α since it increases very slowly with increase of range provided the trajectory is reasonably flat. We shall still denote by n the ratio of the range in vacuum to the range in air, which is now given by $n = 1 + \frac{2}{3} kX$.

38. The Angle of Fall.—Differentiating (28) we get

$$\frac{dv}{dx} = \tan \phi = \tan \alpha - \frac{gx}{V^2 \cos^2 \alpha}(1 + kx) \qquad (29)$$

But the angle of fall (ω) is the negative of the value of ϕ at the point of fall, where $x = X$, hence

$$\tan \omega = -\tan \alpha + \frac{gX}{V^2 \cos^2 \alpha}(1 + kX),$$

and since $\frac{2V^2 \sin \alpha \cos \alpha}{g} = nX$, this may be written

$$\tan \omega = -\tan \alpha + \frac{2\tan \alpha}{n}(1 + kX) = \tan \alpha \frac{2 + 2kX - n}{n}$$

Also from $1 + \frac{2}{3} kX = n$ we get $2kX = 3(n - 1)$, whence

$$\tan \omega = \tan \alpha \left(2 - \frac{1}{n}\right) \qquad (30)$$

From (30) we see that the angle of fall is always greater than the angle of departure, but can never reach double the latter.

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39. The Time of Flight.—Returning to equation (18), and writing $\frac{dx}{dt}$ for u, and x for s, we have

$$\frac{dx}{dt} = V \cos \alpha \,.\, \epsilon^{-kx}$$

Separating the variables and integrating between corresponding limits,

$$\int_0^X \epsilon^{kx} dx = V \cos \alpha \int_0^T dt$$

$$\frac{\epsilon^{kX} - 1}{k} = V \cos \alpha \,.\, T$$

Expanding ϵ^{kX} and neglecting higher powers than the second,

$$X\left(1 + \frac{kX}{2}\right) = V \cos \alpha \,.\, T$$

But from $1 + \frac{2}{3}kX = n$, we get $1 + \frac{kX}{2} = \frac{3n+1}{4}$, therefore

$$T = \frac{3n+1}{4} \cdot \frac{X}{V \cos \alpha} \qquad (31)$$

Since the range in vacuum, *for the same values of V and α*, would be nX, $\frac{nX}{V \cos \alpha}$ is the time of flight in vacuum, and so we see that the time of flight in air is always less than it would be in vacuum, approaching $\frac{3}{4}$ the latter's value as a minimum. Of course $\cos \alpha$ may be taken as unity for flat trajectories.

40. The Striking Velocity.—Equation (18), with the substitution of x for s, gives the value of the horizontal component of the velocity at any point in the trajectory, and since the striking velocity is the horizontal velocity at the point of fall multiplied by sec ω, we have

$$v_\omega = u_\omega \sec \omega = \frac{V \cos \alpha}{\epsilon^{kX} \cos \omega}$$

whence, putting for ϵ^{kX} the first two terms of its expansion and calling $\frac{\cos \alpha}{\cos \omega}$ unity,

$$v_\omega = \frac{V}{1 + kX} = \frac{2}{3n - 1} V. \qquad (32)$$

The striking velocity, therefore, is always less than the initial velocity, being reduced to $\frac{V}{4}$ when $n = 3$, a value it seldom reaches.

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41. The Coordinates of the Vertex.—Since the trajectory is horizontal at its highest point, we obtain the abscissa (x_0) of the vertex by putting $\phi = 0$ in equation (29), thus getting

$$\frac{V^2 \sin 2\alpha}{2g} = x_0(1 + kx_0)$$

and since $\frac{V^2 \sin 2\alpha}{g} = nX$, this may be written,

$$x_0(1 + kx_0) = \frac{nX}{2}$$

a quadratic whose solution gives

$$x_0 = \frac{-1 + \sqrt{1 + 2knX}}{2k}$$

which, since $kX = \frac{8}{3}(n - 1)$, may be written

$$x_0 = \frac{\sqrt{1 + 3n(n-1)} - 1}{3(n-1)} X \qquad (33)$$

Since for $n = \infty$ this reduces to $\frac{X}{\sqrt{3}}$, we see that the abscissa of the vertex is always greater than half the range but never reaches .58X.

The ordinate of the vertex may be obtained by substituting the value of x_0 (eq. 33) for x in the equation to the trajectory, but a much simpler and as accurate a determination is given by

$$y_0 = \frac{gT^2}{8} \qquad (34)$$

This assumes the height of the vertex to be that from which a body would fall freely (in vacuum) in half the time of flight. Actually the vertical velocity of the projectile is reduced by air resistance, but since the time the projectile takes to describe the descending branch of the trajectory is somewhat greater than half the time of flight, the air resistance is approximately allowed for by equation (34).

42. As an example of the use of the foregoing formulæ, we will compute the various elements of the 500-yard trajectory of the United States magazine rifle, for which, with $V = 2000$ f. s., $\alpha = 31' 35''$. Here $n = \frac{V^2 \sin 2\alpha}{gX} = 1.522$.

Then, using ω and α in place of their tangents, which we can do when they are such small angles, $\omega = \left(2 - \frac{1}{n}\right)\alpha = 1.343\,\alpha$; whence $\omega = 42' 24''$.

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Next, to find the time of flight, we have $T = \frac{3n+1}{4} \cdot \frac{X}{V}$ $= \frac{5.566}{4} \times \frac{1500}{2000} = 1.04$ seconds. Then $v_\omega = \frac{2}{3n-1} V = \frac{4000}{3.566}$ $= 1122$ f. s., which is the striking velocity. For the abscissa of the vertex, we have $x_0 = \frac{\sqrt{1+3n(n-1)}-1}{3(n-1)} X = .536\,X = 804$ ft., and, for the ordinate of the vertex, $y_0 = \frac{gT^2}{8} = 4.35$ ft.

The official description of this rifle gives the following values, mostly determined by experiment: $\omega = 45'\ 55''$; $T = 1.012$ seconds; $v_\omega = 1138$ f. s.; $x_0 = 816$ ft.; $y_0 = 4.16$ ft.

43. As another example, take the case of the 6-inch gun with $V = 2400$ f. s. and $\alpha = 2°$, for which angle of departure the range is 3100 yards.

$$\text{Then } n = \frac{2400^2 \sin 4°}{32.2 \times 9300} = 1.342.$$

$$\tan \omega = \left(2 - \frac{1}{n}\right) \tan \alpha = 1.255 \tan \alpha = [\log \tan 8.64172] = 2°\ 31'$$

$$T = \frac{3n+1}{4} \cdot \frac{X}{V} = \frac{5.026}{4} \cdot \frac{9300}{2400} = 4.87 \text{ sec.}$$

$$v_\omega = \frac{2}{3n-1} V = \frac{2 \times 2400}{3.026} = 1586 \text{ f. s.}$$

The official range table for this gun gives $\omega = 2°\ 39'$; $T = 4.8$ sec.; and $v_\omega = 1566$ f. s.

44. **To Find the Ordinate of the Trajectory at a Given Distance from the Gun.**—If the value of n is known, k may be found from $n = 1 + \frac{2}{3} kX$, and then, by substituting the proper value of x in the equation to the trajectory, the corresponding value of y may be computed.

If, however, a range table for the gun is available, the approximate value of y for any value of x may be found much more easily as follows: Referring to Fig 6, let x′ y′ be the coordinates of a point M on the trajectory for which $\alpha = \theta' + p$. Then, by the principle of the rigidity of the trajectory, if the angle of departure were θ', the horizontal range would equal OM, or what is practically the same thing, x′. Consequently, if we take from the range table the angle of departure for a range x′ and subtract it from the angle of departure for the given trajectory, the result will be the angle p. Then $y' = x' \tan p$.

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For example, taking the 4000-yard trajectory of the 6-inch gun of 2400 f. s. I. V., let us find the ordinate at 3500 yards from the gun. Here the angle of departure for 4000 yards range is 2° 52′; that for 3500 yards range is 2° 22′; therefore $y' = x' \tan [2° 52' - 2° 22'] = 3500 \tan 30' = 30.54$ yards.

45. The Danger Space.—By this term is meant an interval of space between the point of fall and the gun, such that the target,

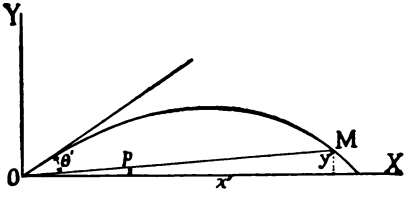

Fig. 6.

if situated at any point within the space, will be hit. In other words, it is the distance from the point of fall to the foot of an ordinate of the trajectory equal in length to the height of the target. Thus, referring to Fig. 7, AH = D is the danger space for a target of height AB = h in the case of the trajectory OBH.

Evidently the method of finding an ordinate of the trajectory by inspection of the range table, as described in 44, will serve equally well to determine the danger space for a given height.

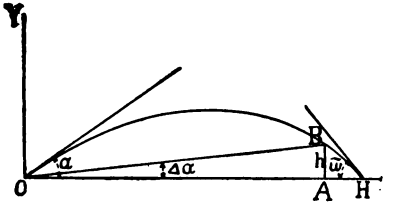

Fig. 7.

A first value of the angle $\Delta\alpha$ is found from the approximate expression $\Delta\alpha = \frac{h}{X}$: then the range corresponding to an angle of departure $\alpha - \Delta\alpha$ is taken, by interpolation if necessary, from the range table, and X minus this range is a first approximation for D. The process being repeated, the second value of $\Delta\alpha$ being found from $\Delta\alpha = \frac{h}{X - D}$, a second, and practically correct, value of D is determined.

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When the value of h is very small in comparison with the range, the danger space is given with sufficient accuracy by the formula

$$D = h \cot \omega \qquad (35)$$

This assumes that the tangent at the point of fall is identical with the curve from H to B (Fig. 7), whereas it really passes above B, so that the result given by (35) is too small.

A more accurate formula for the danger space is deduced as follows: Calling the very small angle AOB $= \Delta\alpha$, we have, very nearly,

$$X\Delta\alpha = D \tan \omega$$

or, since $\Delta\alpha = \frac{h}{X-D}$,

$$\frac{hX}{X-D} = D \tan \omega$$

$$D = h \cot \omega \left(\frac{X}{X-D}\right) = h \cot \omega \left(1 + \frac{D}{X} + \frac{D^2}{X^2} + \ldots\right)$$

whence, neglecting the higher powers of the fraction $\frac{D}{X}$ and putting for D in the expression its approximate value h cot ω

$$D = h \cot \omega \left(1 + \frac{h \cot \omega}{X}\right) \qquad (36)$$

46. As an example, we will compute by each of the foregoing methods the danger space for a target 30 feet high in the case of an 8-inch gun, I. V. = 2000 f. s., at 2000 yards range.

Here $\Delta\alpha = \frac{h}{X} = \frac{30}{6000} = 17'\ 10''$; α by the range table is 1° 34′; and the range for $\alpha - \Delta\alpha = 1°\ 16'\ 50''$ is 1675 yards; thus giving $D_1 = 2000 - 1675 = 325$ yards: then $\Delta\alpha = \frac{10}{1675} = 20'$ 30″, and the range for 1° 13′ 30″ being 1610 yards, we have D = 2000 — 1610 = 390 yards.

Again, $\omega = 1°\ 50'$ (by the range table) and this gives D = h cot ω = 312 yards.

Finally, by equation (36), we have $D = 312 \left(1 + \frac{312}{2000}\right) = 361$ yards.

EXAMPLES.—IV.

(1) The 6-inch gun with V = 2900 f. s. and $\alpha = 1°\ 07'\ 49''$ gives a range of 2625 yards. Find n and thence ω, T and v_ω.

1.308; 1° 24′; 3.34 sec.; 1983 f. s.

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(2) Find the coordinates of the vertex of the trajectory of Example 1; also the danger space for 20 feet height.

$x_0 = 1381$ yds.; $y_0 = 44.9$ ft.
$D = 301$ yds.

(3) The 6-inch gun with $V = 2900$ f. s. and $\alpha = 3^\circ\ 03'\ 51''$ gives a range of 5394 yards. Find n and thence ω, T and v_ω.

1.723; 4° 22′; 8.61 sec.; 1391 f. s.

(4) Find the coordinates of the vertex of the trajectory of example 3; also the danger space for 20 feet height.

$x_0 = 2924$ yds.; $y_0 = 298$ ft.
$D = 89$ yds.

(5) The 4-inch gun of 2900 f. s. I. V. ranges 2600 yds. for an angle of departure of 1° 07′ 26″. Find n and thence ω, T and v_ω.

1.313; 1° 24′; 3.32 sec.; 1973 f. s.

(6) The 4-inch gun of 2900 f. s. I. V. ranges 5166 yds. for an angle of departure of 3° 03′ 47″. Find n and thence ω, T and v_ω.

1.798; 4° 26′; 8.54 sec.; 1320 f. s.

(7) The 4-inch gun of 2900 f. s. I. V. ranges 6599 yds. for an angle of departure of 5° 02′ 58″. Find n and thence ω, T and v_ω.

2.313; 7° 58′; 13.55 sec.; 977 f. s.

(8)* The computed angle of departure for 1000 yds. range for the United States magazine rifle (I. V. = 2000 f. s.) is 1° 33′. Find n and thence ω, T and v_ω.

2.239; 2° 24′; 2.89 sec.; 700 f. s.

(9) The 12-inch gun of 2400 f. s. gives 5900 yds. range for $\alpha = 3^\circ\ 33'$. Find n and thence ω, T and v_ω.

1.331; 4° 44′; 9.20 sec.; 1604 f. s.

(10) The range table of the 3-inch gun of 2800 f. s. I. V. gives $\alpha = 1^\circ\ 01'$ for $X = 2000$ yds., and shows that X changes 100 yds. for each 4′ change of α. Find the ordinates of the 2000-yd. trajectory at 1700, 1800, and 1900 yds. from the gun.

5.93; 4.19; 2.21 yds.

*The approximate formula for the striking velocity having been deduced by neglecting higher powers of kx than the first, it cannot be expected to give very close results in cases like this.

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(11) What is the danger space for a height of 20 feet in the case of the 2000-yd. trajectory of example 9?

338 yds.

(12) The angle of fall for the 2000-yd. trajectory of the 3-inch gun is 1° 26′. Compute the danger space for 20 feet height. 302 yds.

(13) The 13-inch gun with $V = 2300$ f. s. gives 2000 yds. range for $\alpha = 1° 08'$. Find n and thence ω, T and v_ω.

1.083; 1° 13′; 2.77 sec.; 2045 f. s.

(14) The 12-inch gun with $V = 2250$ f. s. gives 2000 yds. range for $\alpha = 1° 11'$. Find n and thence ω, T and v_ω.

1.082; 1° 16′; 2.83 sec.; 2004 f. s.

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CHAPTER V.

THE COMPUTATION AND USE OF BALLISTIC TABLES.

47. The Time Integral.—Returning now to Mayevski's equations (10) for the retardation due to air resistance, and taking the most general expression for that retardation (14), it may be written, after separation of the variables,

$$dt = -\frac{C'}{A} \cdot \frac{dv}{v^n} \qquad (37)$$

in which A and n are Mayevski's constants, and C′ is the reduced ballistic coefficient $\left(\frac{w}{\delta i d^2}\right)$.

Suppose now a projectile to travel so nearly horizontally that its velocity is not affected by gravitation, but is only affected by air resistance: then the integration of (37) between corresponding limits, t_0 and t of t and v_0 and v of v, will give the elapsed time $(t - t_0)$ corresponding to the loss of velocity $(v_0 - v)$. We will assume the velocity at the origin of time to be 3600 f. s., or $t_0 = 0$, $v_0 = 3600$, and we will designate by T_v the elapsed time from the origin till the velocity is reduced to v. Thus we have

$$\int_0^{T_v} dt = -\frac{C'}{A}\int_{3600}^{v} \frac{dv}{v^n} \qquad (38)$$

$$T_v = \frac{C'}{A}\frac{1}{n-1}\left(\frac{1}{v^{n-1}} - \frac{1}{3600^{n-1}}\right) \qquad (39)$$

Substituting in (39) the values of A and n given in the first of Mayevski's equations (10), we get

$$T_v = C'\left(\frac{[2.6505893]}{v^{0.55}} - 4.9502\right) \qquad (40)$$

and the value of this, computed for any value of v between 3600 and 2600, is the time in seconds which it takes for air resistance to reduce the velocity of a projectile whose reduced ballistic coefficient is C′ from 3600 f. s. to v f. s.

Below 2600 f. s. the values of A and n change to those given in the second of Mayevski's equations (10), and with the new values the limits of integration change and we have

$$\int_{T_{2600}}^{T_v} dt = -\frac{C'}{A}\int_{2600}^{v} \frac{dv}{v^n}$$

$$T_v = T_{2600} + \frac{C'}{A}\frac{1}{n-1}\left(\frac{1}{v^{n-1}} - \frac{1}{2600^{n-1}}\right)$$

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which, after substituting the values of A and n and of T_{2000} (calculated from (40)), reduces to

$$T_v = C'\left(\frac{[3.0587042]}{v^{0.7}} - 3.6880\right) \tag{41}$$

and the value of this, computed for any value of v between 2600 and 1800, is the time in seconds it takes the air resistance to reduce the velocity of a projectile (whose reduced ballistic coefficient is C') from 3600 f. s. to v f. s.

Proceeding in the same manner with the other values of A and n, each integration being performed between the limits which correspond to the particular values of A and n used, we obtain the following expressions for T_v

$$\left.\begin{aligned}
&\textit{v between 3600 f. s. and 2600 f. s.}\\
&T_v = C'\left(\frac{[2.6505893]}{v^{0.55}} - 4.9502\right)\\
&\textit{v between 2600 f. s. and 1800 f. s.}\\
&T_v = C'\left(\frac{[3.0587042]}{v^{0.7}} - 3.6880\right)\\
&\textit{v between 1800 f. s. and 1370 f. s.}\\
&T_v = C'\left(\frac{[3.8807404]}{v} - 1.8837\right)\\
&\textit{v between 1370 f. s. and 1230 f. s.}\\
&T_v = C'\left(\frac{[6.7180677]}{v^2} + 0.8790\right)\\
&\textit{v between 1230 f. s. and 970 f. s.}\\
&T_v = C'\left(\frac{[12.5960688]}{v^4} + 2.6089\right)\\
&\textit{v between 970 f. s. and 790 f. s.}\\
&T_v = C'\left(\frac{[6.9255270]}{v^2} - 1.8801\right)\\
&\textit{v less than 790 f. s.}\\
&T_v = C'\left(\frac{[4.3301086]}{v} - 15.4595\right)
\end{aligned}\right\} \tag{42}$$

48. The Space Integral.—Multiplying each side of equation (37) by v, we have $vdt = -\frac{C'}{A} \cdot \frac{dv}{v^{n-1}}$, or putting ds for vdt,

$$ds = -\frac{C'}{A} \cdot \frac{dv}{v^{n-1}} \tag{43}$$

and the integration of this equation between corresponding limits, s_0 and s of s and v_0 and v of v, will give the space traversed

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$(s - s_0)$ while the velocity is reduced by air resistance from v_0 to v, supposing as in 47 that the path of the projectile is so nearly horizontal that the effect of gravitation on the velocity in the trajectory may be neglected. We will assume that the origin of space, as well as the origin of time, coincides with the value 3600 of v, or $s_0 = 0$, $v_0 = 3600$, and we will designate by S_v the space passed over from the origin to the point where the velocity is reduced to v. Thus we have

$$\int_0^{S_v} ds = -\frac{C'}{A} \int_{3600}^{v} \frac{dv}{v^{n-1}}$$

$$S_v = \frac{C'}{A} \frac{1}{n-2}\left(\frac{1}{v^{n-2}} - \frac{1}{3600^{n-2}}\right) \qquad (44)$$

Substituting in (44) the values of A and n given in the first of Mayevski's equations (10) we get

$$S_v = C'(21780.89 - [2.7377395]\, v^{0.45})$$

and the value of this, computed for any value of v between 3600 and 2600, is the space in feet passed over by a projectile (whose reduced ballistic coefficient is C') while air resistance reduces its velocity from 3600 f. s. to v f. s.

Proceeding, exactly as in the case of the time integrals, with the other values of A and n, we obtain the following expressions for S_v:

v between 3600 f. s. and 2600 f. s.

$$S_v = C'\,(21780.89 - [2.7377395]\, v^{0.45})$$

v between 2600 f. s. and 1800 f. s.

$$S_v = C'\,(31227.10 - [3.4266809]\, v^{0.3})$$

v between 1800 f. s. and 1370 f. s.

$$S_v = C'\,(62875.33 - [4.2429561] \log v)$$

v between 1370 f. s. and 1230 f. s.

$$S_v = C'\left(\frac{[7.0190977]}{v} + 365.69\right)$$

v between 1230 f. s. and 970 f. s.

$$S_v = C'\left(\frac{[12.7210075]}{v^3} + 6034.48\right)$$

v between 970 f. s. and 790 f. s.

$$S_v = C'\left(\frac{[7.2265570]}{v} - 5571.36\right)$$

v less than 790 f. s.

$$S_v = C'\,(158436.8 - [4.6923243] \log v) \qquad (46)$$

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49. The Use of T_v and S_v.—T_v and S_v are known respectively as the *time function* and the *space function,* and their values have been computed, for values of v from 3600 to 500, on the supposition that C′ is unity, and will be found in Table I appended, under the headings T_z and S_z with z as the argument. To apply them to any projectile it is only necessary to multiply the tabular values by the particular value of C′ belonging to that projectile.

The use of the tabulated values of T_z and S_z is best illustrated by an example. Suppose we wish to find how long it takes the air resistance to reduce the velocity of a 6-inch 100-pound projectile from 2100 f. s. to 1900 f. s., and how far the projectile travels while its velocity is being thus reduced. Entering Table I with $z = 1900$ as an argument, we find $T_{1900} = 2.114$ and $S_{1900} = 5505.0$, which means that a projectile for which $C' = 1$ would have its velocity reduced from 3600 f. s. to 1900 f. s in 2.114 seconds and would travel 5505.0 feet meanwhile. Again entering the table, with $z = 2100$ as an argument, we find $T_{2100} = 1.721$ and $S_{2100} = 4720.9$, which means that a projectile for which $C' = 1$ would have its velocity reduced from 3600 f. s. to 2100 f. s. in 1.721 seconds and would travel 4720.9 feet meanwhile. Evidently, then, the difference of the two foregoing values of T_z, $T_{1900} - T_{2100} = 2.114 - 1.721 = 0.393$ seconds, is the time in which the velocity is reduced from 2100 f. s. to 1900 f. s., and so too the difference between the two values of S_z $S_{1900} - S_{2100} = 5505.0 = 4720.9 = 784.1$ feet, is the space traversed while the velocity is being reduced from 2100 f. s. to 1900 f. s. This, however, is for a projectile whose ballistic coefficient is unity, and since, in this particular case of a 6-inch projectile, $C' = \frac{w}{\delta i d^2} = \frac{100}{36}$ $(\delta = 1;\ i = 1)$, we have $T = \frac{100}{36} \times 0.393 = 1.092$ seconds and $S = \frac{100}{36} \times 784.1 = 2178.1$ feet as the actual elapsed time and space traversed by the 6-inch projectile while its velocity is reduced by air resistance from 2100 f. s. to 1900 f. s.

The foregoing procedure is mathematically expressed by the formulæ:

$$T = C'(T_{v_2} - T_{v_1}) \qquad (47)$$

$$S = C'(S_{v_2} - S_{v_1}) \qquad (48)$$

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50. By means of equations (47) and (48), if any two of the four quantities T, S, v_1 and v_2 are known, the other two can be found, as illustrated in the following examples:

A standard 10-inch projectile, weight 500 pounds, has an initial velocity of 2090 f. s.; what is its velocity after 3 seconds flight in air of standard density? Here $C' = \frac{500}{100} = 5$, $T = 3$ and $v_1 = 2000$. Therefore $T_{v_2} = \frac{T}{C'} + T_{2000} = \frac{3}{5} + 1.909 = 2.509$. Whence, from Table I, $v_2 = 1729.6$ f. s.

A standard 12-inch projectile, weight 850 pounds, has a velocity of 1800 f. s. after travelling 1000 yards in air of standard density; what was its initial velocity? Here $C' = \frac{850}{144} = 5.903$, $S = 3000$ and $v_2 = 1800$. Therefore $S_{v_1} = S_{1800} - \frac{S}{C'} = 5918.8 - 508.2 = 5410.6$. Whence, by Table I, $v_1 = 1923$ f. s.

A standard 8-inch projectile, weight 250 pounds, starting with 2400 f. s., travels for 2 seconds in air of standard density: how far did it go? Here we must use both (47) and (48). First, from (47) we find v_2, as follows: $T_{v_2} = \frac{T}{C'} + T_{2400} = \frac{2}{3.906} + 1.239 = 1.751$; whence $v_2 = 2083$ f. s. Then, from (48), we find S, as follows: $S = C' (S_{2083} - S_{2400}) = 3.906 (4785.5 - 3637.6) = 4483.6$ feet $= 1495$ yds.

51. Limits of Accuracy.—The foregoing methods are, of course, only strictly applicable to such parts of trajectories as may, without material error, be considered horizontal straight lines, since in deducing (47) and (48) we have entirely neglected the effect of gravitation. They will give sufficiently accurate results when applied to any arc of a trajectory if the length of that arc is not materially greater than the length of its chord and the latter's inclination to the horizontal is not more than 10° or 15°. But they are principally applied to the entire trajectories of guns fired with angles of departure not exceeding 3° or 4°, giving the striking velocity and time of flight for ranges as great as 5000 yards, in the case of medium and large guns of high velocity, with as much accuracy as is obtainable by any other method of computation.

52. The Correction for the Density of the Air.—As already stated, the resistance of the air is proportional to its density, or,

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what is an equivalent, to the weight of unit volume of the air, and since Mayevski's experimentally determined values of the constant A are for a standard atmospheric condition, in which the temperature is 15° C., the barometric height 750 mm. and the humidity half that of saturation, we must take account of any different state of the atmosphere by multiplying the values of A by a constant δ, which is the ratio of the actual density of the air to the assumed standard density. This is most conveniently done by introducing δ as a factor in the denominator of the ballistic coefficient, and accordingly, whenever the density of the air differs from the standard, we use the reduced ballistic coefficient $C' = \frac{w}{\delta i d^2}$.

The weight of a cubic foot of air of the standard density is 526.9 grains, and Table II, appended, gives the value of δ for any temperature from 0° to 100° F. and for any barometric height from 28 to 31 inches, it being assumed that in all cases the air is half saturated with moisture.

As an example, suppose, in the first case under 50, the temperature was 51° and the barometer stood at 30.12; then from Table II, by interpolation, we find $\delta = 1.037$; whence $C' = \frac{w}{\delta i d^2} = \frac{500}{1.037 \times 100} = 4.8216$; $T_{v_2} = \frac{T}{C'} + T_{2000} = \frac{3}{4.8216} + 1.909 = 2.531$; and so $v_2 = 1721.1$ f. s. Thus the greater density of the air at 51° and 30.12 inches than of standard air results in a loss of velocity (during 3 seconds' flight) greater by 8.5 f. s. than would be the case in standard air.

53. The Coefficient of Form.—Since the resistance of the air depends upon the shape of the projectile, it is necessary to introduce a *coefficient of form* (i) into the ballistic coefficient to take account of any difference of shape from that of the standard projectiles. The experiments upon which our values of A and n are based were made with projectiles about three calibers long having ogival heads of two calibers radius, and for such projectiles the value of i is unity; for projectiles to which the air offers less resistance, i is less than one, and for those to which it offers greater resistance, i is greater than one.

The value of i depends to some extent upon the smoothness of surface and the length of the projectile, also upon the form of its rear end, but it depends primarily upon the shape of head.

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Unsteadiness of flight, due to insufficient rotation, greatly increases i, and a rounded form of base would probably considerably diminish it, at least at moderate velocities. The ogival form of head appears to be the most favorable, and is in universal use, the radius of the ogive varying from $1\frac{1}{2}$ to 3 calibers in different designs. French experiments indicate that with this form of head the resistance is approximately proportional to the sine of the semi-angle of the point, and upon this basis the values of i for ogives of different radii would be as follows:

Radius	i
1 cal.	1.309
$1\frac{1}{2}$ "	1.127
2 "	1.000
$2\frac{1}{2}$ "	.907
3 "	.836

Apparently the form of head near its junction with the cylindrical body is the most important factor in determining the resistance to projectiles, and cutting off the extreme point, or fitting on the point a blunt-nosed cap, adds but little to the air resistance; if, however, the diameter of the flattened part approaches the caliber, the resistance is greatly increased, and with a perfectly flat head the value of i probably exceeds 2.

54. The value of i for any particular form of projectile is determined experimentally by firing through screens at a known distance apart and comparing the actual loss of velocity with the loss of velocity given by the ballistic tables for a standard projectile. Thus, putting $\frac{w}{\delta i d^2}$ for C′ in (48), we have

$$S = \frac{w}{\delta i d^2} (S_{v_2} - S_{v_1})$$

whence

$$i = \frac{w}{\delta d^2} \cdot \frac{S_{v_2} - S_{v_1}}{S} \qquad (49)$$

The following is an example of the use of (49): A Krupp 28 cm. (11″.024) gun fired a 345 kilo. (760.4 lb.) projectile through screens at 100 and 2000 meters, respectively, from the gun, giving measured velocities of 516.5 m. s. (1694.6 f. s.) and

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452.1 m. s. (1483.3 f. s.). The value of δ at the time of firing being 1.013, to find the value of i for the projectile. Here S = 1900 m. = 6233.7 ft.; $S_{v_2} = 7389.5$; $S_{v_1} = 6377.3$; and so $i = \frac{760.4 \times 1012.2}{1.013 \times 11.024^2 \times 6233.7} = 1.0029$. Actually i should come out unity, as the projectile was similar to those used in the experiments to determine A and n, but the unavoidable errors in measurements of velocity, slight variations in the projectiles themselves, the effect of wind and of the variable density of the air in the path of the projectile, all produce variations in the experimentally determined values of i, so that only the mean of a great number of determinations can safely be considered reliable.

The value of i can also be determined by a comparison of actual with calculated ranges, but the causes of variation in the range are so numerous that nothing less than consistent results given by many series of shots should cause the acceptance of any other value than unity for the coefficient of form of any projectile which does not greatly depart from the standard.

55. The Muzzle Velocity.—What is called the *muzzle* or *initial velocity* is determined by measuring the velocity at a known distance from the gun and then, by means of Table I, calculating what the velocity would have been at the muzzle if in the projectile's passage thence to the velocity screens it had been opposed by an air resistance corresponding to its velocity through still air. Actually, however, the projectile immediately after leaving the gun is surrounded by gases moving even more rapidly than itself, and experiments have shown that the velocity increases, instead of falling off, during a travel which, in the case of large guns, is as much as 50 yards. Thus, the *muzzle velocity* is really a fictitious quantity, not actually existent when the projectile leaves the gun, but such as the projectile would have to be projected from the muzzle into still air with in order to describe its actual trajectory.

EXAMPLES.—V.

(1) The initial velocity of the 3-inch 13-pound projectile is 2800 f. s. What is the elapsed time till its velocity is 2600 f. s.; 2400 f. s.; 2200 f. s.; in a standard atmosphere?

0.341; 0.729; 1.176 sec.

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(2) What is the space traversed while the projectile of example 1 has its velocity reduced to 2600 f. s.; 2400 f. s.; 2200 f. s.?

922; 1890; 2917 ft.

(3) The 12-inch 850-pound projectile has an I. V. of 2400 f. s. What are the remaining velocities at 1000, 2000, and 3000 yds. range in a standard atmosphere? What if the temperature is 82° F. and the barometer 29.2 inches?

2256; 2118; 1986 f. s.
2264; 2133; 2008 f. s.

(4) What are the times of flight in the second case of example 3?

1.283; 2.648; 4.094 sec.

(5) How long does it take a 6-inch 100-pound projectile with 2000 f. s. I. V. to travel 1000, 2000, 3000 yards, and what are the remaining velocities, the temperature being 44° and the barometer 30.15?

1725; 1485; 1282 f. s.
1.614; 3.493; 5.666 sec.

(6) A Krupp 15 cm. (5".91) gun, w = 112.2 lbs., V = 1667.6 f. s., gave as the mean of 10 shots a measured velocity of 1656 f. s. at 50 m. (164 ft.) and 1358 f. s. at 1500 m. (4921 ft.) from the gun. Compute the values of the velocities at those ranges.

1656 f. s.; 1363 f. s.

(7) The same gun, d = 5".91, w = 112.2 lbs., with V = 1763.6 f. s., gave a measured velocity of 1740.7 f. s. at 100 m. (328 ft.) and 1369 f. s. at 2000 m. (6562 ft.). Compute the values of the velocities at those ranges.

1740 f. s.; 1348 f. s.

(8) A Krupp 30.5 cm. (12") gun, w = 1001 lbs., V = 1721.7 f. s., gave measured velocities 1711 f. s. at 100 m. (328 ft.), 1692 f. s. at 300 m. (984 ft.) and 1518 f. s. at 2000 m. (6562 ft.). Compute the values of the velocities at those ranges.

1711 f. s.; 1690 f. s.; 1521 f. s.

(9) The 4".72 gun, w = 45 lbs., gave a measured velocity of 2204 f. s. at 175 feet from the muzzle, thermometer 65°, barometer 30.43 inches. What was the muzzle velocity?

2228 f. s.

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(10) A 12″ projectile, w = 850 lbs., was fired through screens, the first 100 feet and the second 200 feet from the muzzle, and the time of flight between screens was 0.047 seconds. What was the muzzle velocity?

2135 f. s.

(11) A Krupp 28 cm. (11″.024) gun, w = 760.4 lbs., gave measured velocities of 532.1 m. s. (1746 f. s.) at 100 m. s. (328 ft.) and 466.1 m. (1529 f. s.) at 2000 m. (6562 ft.) from the gun, δ being 1.013. Compute the value of i.

.9992.

(12) A flat headed 6-inch projectile of 70 lbs. weight, when fired through screens 150 feet apart, gave measured velocities 1881 f. s. and 1835 f. s. at the screens. What is the value of i?

2.460.

(13) A Krupp 24 cm. (9″.45) gun, w = 474 lbs., gave measured velocities of 524 m. (1719 f. s.) at 100 m. (328 ft.) and 445 m. s. (1460 f. s.) at 2000 m. (6562 ft.), the value of δ being 1.060. What is the value of i?

.9969.

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CHAPTER VI.

DIFFERENTIAL EQUATIONS OF THE TRAJECTORY IN AIR. SIACCI'S METHOD.

56. From the hypothesis, already made in **28**, that the resultant air resistance acts in the line of the projectile's axis, which itself coincides with the tangent to the trajectory, it follows that the trajectory is plane. For, if a vertical plane be passed through any element of the trajectory, that plane will contain the two forces which alone act upon the projectile, namely, gravity and the resistance of the air, and so their resultant will lie in the same plane and the consecutive element must also lie in it. Hence, all the elements of the trajectory lie in the same vertical plane.

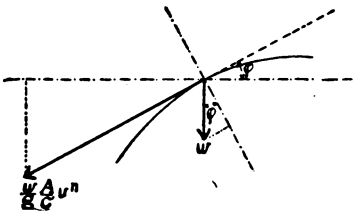

Fig. 8.

Figure 8 represents a portion of the trajectory, with the two forces acting upon the projectile, its weight (w) acting vertically, and the resistance of the air $\left(\frac{w}{g}\frac{A}{C'}v^n\right)$ acting in the tangent. Kinetic equilibrium results from the balancing of these two forces by the inertia forces, $\frac{w}{g}\frac{dv}{dt}$ acting in the tangent and $\frac{w}{g}\frac{v^2}{\rho}$ acting in the normal.

57. The Differential Equations.—Resolving forces along the normal to the trajectory, the inertia force $\frac{w}{g}\frac{v^2}{\rho}$, commonly called the centrifugal force, must balance the resolved part of w: thus we have,—

$$\frac{w}{g}\frac{v^2}{\rho} = w\cos\phi$$

$$\frac{v^2}{\rho} = g\cos\phi \qquad (50)$$

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In other words the acceleration towards the center of curvature is the resolved part of g in that direction.

But the radius of curvature ρ equals $-\frac{ds}{d\phi}$; hence

$$v^2 d\phi = -g \cos\phi ds = -gdx$$

$$gdx = -v^2 d\phi \qquad (51)$$

Dividing each side of (51) by dt, and putting u for $\frac{dx}{dt}$,

$$gu = -v^2 \frac{d\phi}{dt}$$

$$gdt = -\frac{v^2 d\phi}{u} = -\frac{v^2 d\phi}{v \cos\phi}$$

$$gdt = -v \sec\phi d\phi \qquad (52)$$

Putting $\cot\phi dy$ for dx in (51), we get

$$g \cot\phi dy = -v^2 d\phi.$$

$$gdy = -v^2 \tan\phi d\phi. \qquad (53)$$

Putting $\cos\phi ds$ for dx in (51), we get

$$g \cos\phi ds = -v^2 d\phi$$

$$gds = -v^2 \sec\phi d\phi \qquad (54)$$

Now resolving horizontally, since the horizontal component of the air resistance $\left(\frac{w}{g}\frac{A}{C'}v^n \cos\phi\right)$ is the only force which acts to produce the horizontal acceleration $\frac{d^2x}{dt^2} = \frac{du}{dt}$, we have

$$\frac{du}{dt} = \frac{d(v\cos\phi)}{dt} = -\frac{A}{C'}v^n \cos\phi$$

But, from (52), $gdt = -v \sec\phi d\phi$, and so this becomes

$$gd(v \cos\phi) = \frac{A}{C'}v^{n+1} d\phi \qquad (55)$$

58. Equations (51), (52), (53), (54) and (55), which are grouped together in (56), are the differential equations of the trajectory, and could the first of (56) be integrated, thus giving a finite relation between v and ϕ, one of those variables could be eliminated from the other equations of (56) and then the values

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of x, t, y and s could all be obtained either exactly or by quadrature.

$$\left.\begin{aligned} gd(v\cos\phi) &= \frac{A}{C'}\,v^{n+1}d\phi \\ gdx &= -v^2 d\phi \\ gdt &= -v\sec\phi d\phi \\ gdy &= -v^2\tan\phi d\phi \\ gds &= -v^2\sec\phi d\phi \end{aligned}\right\} \qquad (56)$$

Since we cannot integrate the first of (56), it is necessary to resort to methods of approximation, and the following method, due to Major F. Siacci of the Italian army, has been generally adopted by artillerists on account of its simplicity and readiness of application.

59. Siacci's Method.—Taking rectangular axes in the plane of fire; origin at the gun; X, as always, horizontal and plus in the direction of projection; Y, as always, vertical and plus upward; let V be the initial velocity and α the angle of departure, and let v be the velocity at any point P of the trajectory. Then, resolving v vertically and parallel to its original direction, and designating by z the component parallel to the initial velocity, we have

$$z = v\cos\phi\sec\alpha \qquad (57)$$

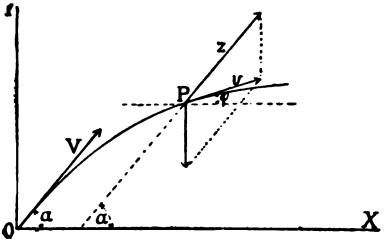

Fig. 9.

This quantity z, which is represented in Fig. 9, is known as the *pseudo-velocity,* and it will readily be seen that it is equal to the actual velocity at the origin where $\phi = \alpha$, and again at a point in the descending branch of the trajectory where $\phi = -\alpha$. At the vertex, where $\phi = 0$, the pseudo-velocity differs most from the true velocity, its value there being v sec α, but if α is small the difference is very small, and for flat trajectories (α not greater

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than about 5°) z may be taken as equal to v throughout without material error.

60. Substituting $z \cos \alpha$ for $v \cos \phi$ and $\frac{z \cos \alpha}{\cos \phi}$ for v in the first of (56), we get

$$g dz = \frac{A}{C'} \frac{\cos^n \alpha}{\cos^{n+1} \phi} z^{n+1} d\phi$$

$$\frac{C'g}{A} \frac{dz}{z^{n+1}} = \frac{\cos^n \alpha}{\cos^{n-1} \phi} \sec^2 \phi d\phi \qquad (58)$$

Now the value of a definite integral remains unchanged when for any variable under the integral sign is substituted a constant which is the mean value of that variable between the limits of integration. Thus, for the variable quantiy * $\frac{\cos^{n-2} \alpha}{\cos^{n-1} \phi}$ we can substitute a constant β, thereby enabling (58) to be integrated, without introducing any error, provided we assign to β its correct value. Leaving the determination of β for future consideration, we have

$$\frac{C'g}{A} \int_z^v \frac{dz}{z^{n+1}} = \beta \cos^2 \alpha \int_\phi^\alpha \sec^2 \phi d\phi \qquad (59)$$

which, putting $-\frac{2g}{A} \int \frac{dz}{z^{n+1}} = I(z)$, becomes, after integration,

$$\tan \alpha - \tan \phi = \frac{C'}{2\beta \cos^2 \alpha} [I(z) - I(V)]$$

$$\tan \phi = \tan \alpha - \frac{C'}{2\beta \cos^2 \alpha} [I(z) - I(V)] \qquad (60)$$

61. Next putting $\frac{z \cos \alpha}{\cos \phi}$ for v in the second of (56), we have

$$g dx = -z^2 \cos^2 \alpha \sec^2 \phi d\phi$$

or, substituting for $\sec^2 \phi d\phi$ its value from (58),

$$g dx = -\frac{C'}{\beta} \cdot \frac{g}{A} \frac{dz}{z^{n-1}}$$

$$\int_0^x dx = -\frac{C'}{\beta} \int_v^z \frac{1}{A} \frac{dz}{z^{n-1}} \qquad (61)$$

$$x = \frac{C'}{\beta} [S(z) - S(V)] \qquad (62)$$

in which S(z) stands for $-\frac{1}{A} \int \frac{dz}{z^{n-1}}$.

*The constant factor $\cos^{n-2} \alpha$ is taken out with the variable factor $\frac{1}{\cos^{n-1} \phi}$ for the reason that their product, the mean value of which we call β, differs very little from unity.

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62. Proceeding in exactly the same way with the third of (56), we get

$$g dt = -z \cos \alpha \sec^3\phi d\phi = -\frac{C'}{\beta \cos \alpha} \frac{g}{A} \frac{dz}{z^n}$$

$$\int_0^t dt = -\frac{C'}{\beta \cos \alpha} \int_v^z \frac{1}{A} \frac{dz}{z^n}$$

$$t = \frac{C'}{\beta \cos \alpha} [T(z) - T(V)] \qquad (63)$$

in which $T(z)$ stands for $-\int \frac{1}{A} \frac{dz}{z^n}$.

63. Now, multiplying (60) by the differential of (62), and putting dy for $\tan \phi \, . \, dx$, we have

$$dy - \tan \alpha \, dx = \frac{C'^2}{2\beta^2 \cos^3 \alpha} [I(z) - I(V)] \frac{1}{A} \frac{dz}{z^{n-1}}$$

which, putting $-\frac{1}{A}\int I(z) \frac{dz}{z^{n-1}} = A(z)$,* becomes, after integration,

$$y - x \tan \alpha = -\frac{C'^2}{2\beta^2 \cos^2 \alpha} [A(z) - A(V) - I(V)\{S(z) - S(V)\}]$$

and finally, dividing by (62), we get

$$\frac{y}{x} = \tan \alpha - \frac{C'}{2\beta \cos^2 \alpha} \left[\frac{A(z) - A(V)}{S(z) - S(V)} - I(V)\right] \qquad (64)$$

64. The Ballistic Formulae.—Equations (60), (62), (63), (64), together with (57), constitute the *ballistic formulæ*. They express the tangent of the inclination, the two co-ordinates, the time and the velocity at any point of the trajectory directly as functions of a new variable, the pseudo-velocity (z), the reduced ballistic coefficient, the angle of departure and the initial velocity being supposed known. In order to determine the elements of any trajectory by means of these formulæ the values of the functions $S(z)$, $T(z)$, $I(z)$ and $A(z)$ for particular values of z must be known, and so their values have been computed for values of z, ranging from 3600 to 500, and will be found in Table I under the respective headings, S_z, T_z, I_z and A_z, with the argument z.

* $A(z)$ means the A function of z, otherwise known as the altitude function, the A having, of course, nothing in common with the constant A on the other side of the equation.

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65. The Time Function.—As shown by (63), the value of the definite integral $-\frac{1}{A}\int_V^z \frac{dz}{z^n}$ $\left(\text{when multiplied by } \frac{C'}{\beta \cos \alpha}\right)$ measures the time of flight from a point where the pseudo-velocity is V to one where it is z. We have represented this definite integral by T(z)—T(V), and we require a table from which its value, for any given values of V and z, may be taken. But, as explained in Chapter V, the values of T_z in Table I were calculated by the formula $T_z = -\frac{1}{A}\int_{3600}^{z} \frac{dz}{z^n}$, so that T_z is the same thing as what we now designate by T(z)—T(3600). Consequently we always have $T(z_1) - T(z_2) = T_{z_1} - T_{z_2}$ and the tabulated *time functions* (T_z) may be used in (63) exactly as they are used in (47), except that now z is the pseudo-velocity and not the real velocity.

66. The Space Function.—In the same way, in (62), the value of the definite integral $-\frac{1}{A}\int_V^z \frac{dz}{z^{n-1}}$, which we have represented by S(z)—S(V), measures $\left(\text{when multiplied by } \frac{C'}{\beta}\right)$ the horizontal space traversed while the pseudo-velocity changes from V to z, and, as the values of S_z in Table I were calculated by the formula $S_z = -\frac{1}{A}\int_{3600}^{z} \frac{dz}{z^{n-1}}$, S_z is what we now designate S(z)—S(3600). Consequently we always have $S(z_1) - S(z_2) = S_{z_1} - S_{z_2}$ and the tabulated *space functions* (S_z) may be used in (62) exactly as they are used in (48), except that now z is the pseudo-velocity and not the real velocity.

67. The Inclination Function.—As shown by (60), the value of the definite integral $-\frac{2g}{A}\int_V^z \frac{dz}{z^{n+1}}$ $\left(\text{when multiplied by } \frac{C'}{2\beta \cos^2 \alpha}\right)$ measures the change in the tangent of the inclination of the trajectory from the point where the pseudo-velocity is V to the point where it is z. We have represented this definite integral by I(z)—I(V), and we require a table from which its value, for any given values of V and z, may be taken. To supply this, the values of $-\frac{2g}{A}\int_{\infty}^{z} \frac{dz}{z^{n+1}}$ have been computed, for values of z from 3600 to 500, and will be found in Table I appended under the heading I_z with z as the argument. Then, as in the case of the time and space functions, we always have $I(z_1) - I(z_2) =$

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$I_{z_1} - I_{z_2}$ and the tabulated *inclination functions* (I_z) may be used in (60) in place of the functions there written I(z).*

The equations from which I_z is calculated are obtained by substituting the successive values of A and n in $I_z = -\frac{2g}{A}\int\frac{dz}{z^{n+1}}$ and integrating, the first integration being between ∞ and z (z from 3600 to 2600), the second being between 2600 and z (z from 2600 to 1800), the third being between 1800 and z (z from 1800 to 1370), and so on, the results being as follows:

$$\left.\begin{array}{l}
\textit{z between 3600 f. s. and 2600 f. s.} \\
\quad I_z = \dfrac{[4.0089663]}{z^{1.55}} \\
\textit{z between 2600 f. s. and 1800 f. s.} \\
\quad I_z = \dfrac{[4.4816993]}{z^{1.7}} + 0.004520 \\
\textit{z between 1800 f. s. and 1370 f. s.} \\
\quad I_z = \dfrac{[5.3880564]}{z^{2}} + 0.017757 \\
\textit{z between 1370 f. s. and 1230 f. s.} \\
\quad I_z = \dfrac{[8.3503224]}{z^{3}} + 0.060830 \\
\textit{z between 1230 f. s. and 970 f. s.} \\
\quad I_z = \dfrac{[14.3075048]}{z^{5}} + 0.109117 \\
\textit{z between 970 f. s. and 790 f. s.} \\
\quad I_z = \dfrac{[8.5577817]}{z^{3}} - 0.050275 \\
\textit{z less than 790 f. s.} \\
\quad I_z = \dfrac{[5.8374246]}{z^{2}} - 0.419591
\end{array}\right\} \quad (65)$$

68. The Altitude Function.—As explained in **63**, A(z) — A(V) stands for the value of the definite integral,

$$-\frac{1}{A}\int_V^z I(z)\frac{dz}{z^{n-1}},$$

* It would have been as well to have tabulated the values of $-\frac{2g}{A}\int_{3600}^{z}\frac{dz}{z^{n+1}}$, making $I_{3600} = 0$ instead of $I_{3600} = .03138$, and thus making the series of values of the inclination functions, like those of the space, time and altitude functions, begin at an imagined origin where z = 3600.

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or (substituting for I(z) its value $-\frac{2g}{A}\left[\frac{dz}{z^{n+1}}=\frac{2g}{Anz^n}\right)$ A(z)—A(V)$=-\frac{2g}{A^2n}\int_V^z\frac{dz}{z^{2n-1}}$, and in order that the value of this integral may be found for any given values of V and z, the values of $-\frac{2g}{A^2n}\int_{3600}^{z}\frac{dz}{z^{2n-1}}$ have been computed, for values of z from 3600 to 500, and will be found in Table I appended under the heading A_z with z as argument. Then, as in the case of the space, time, and inclination functions, we always have $A(z_1)-A(z_2)=A_{z_1}-A_{z_2}$, and the tabulated *altitude functions* (A_z) may be used in (64) in place of the functions there written A(z).

The equations from which A_z is calculated are obtained by substituting the successive values of A and n in $A_z=-\frac{2g}{A^2n}\int\frac{dz}{z^{2n-1}}$ and integrating, exactly as was done in 47, where the values of T_v were being found. The results are as follows:

z between 3600 f. s. and 2600 f. s.

$$A_z=\frac{[6.3585256]}{z^{1.1}}-279.636$$

z between 2600 f. s. and 1800 f. s.

$$A_z=\frac{[7.2393734]}{z^{1.4}}-[1.0817886]\,z^{0.3}-39.264$$

z between 1800 f. s. and 1370 f. s.

$$A_z=\frac{[8.9677668]}{z^3}-[2.4923277]\log z+1051.999$$

z between 1370 f. s. and 1230 f. s.

$$A_z=\frac{[14.7673601]}{z^4}+\frac{[5.8032141]}{z}-57.984$$

z between 1230 f. s. and 970 f. s.

$$A_z=\frac{[26.6025436]}{z^9}+\frac{[11.7589003]}{z^3}+329.618$$

z between 970 f. s. and 790 f. s.

$$A_z=\frac{[15.1822787]}{z^4}-\frac{[5.9279108]}{z}+624.046$$

z less than 790 f. s.

$$A_z=\frac{[9.8665032]}{z^3}+[4.3151504]\log z-68192.39$$

(66)

NOTE.—It should be explained that the values of I(z) to be substituted in the integral expression for the value of A(z) — A(V) must include the constants of integration whose values are given in (65), so that the statement that the values of A_z result from the integration of $\frac{2g}{A^2n}\int\frac{dz}{z^{2n-1}}$ is inaccurate.

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69. The ballistic formulæ, in which hereafter the tabular functions T_z, S_z, I_z and A_z will be substituted for $T(z)$, $S(z)$, $I(z)$ and $A(z)$, are grouped together in (67).

$$\left.\begin{aligned} x &= \frac{C'}{\beta}[S_z - S_v] \\ \frac{y}{x} &= \tan\alpha - \frac{C'}{2\beta\cos^2\alpha}\left[\frac{A_z - A_v}{S_z - S_v} - I_v\right] \\ \tan\phi &= \tan\alpha - \frac{C'}{2\beta\cos^2\alpha}[I_z - I_v] \\ t &= \frac{C'}{\beta\cos\alpha}[T_z - T_v] \\ v &= z\cos\alpha\sec\phi \end{aligned}\right\} \quad (67)$$

70. **The Value of β.**—As explained in 60, β is the mean value of $\frac{\cos^{n-2}\alpha}{\cos^{n-1}\phi}$ between the limits of integration. Thus, for $n = 2$, β is the mean value of sec. ϕ, which decreases from sec α at the point of departure to unity at the vertex and then increases to sec ω. For $n = 3$, β is the mean value of $\cos\alpha\sec^2\phi$, which decreases from sec α at the point of departure to $\cos\alpha$ at the vertex and then increases to $\cos\alpha\sec^2\omega$ at the point of fall. For $n = 5$, β is the mean value of $\cos^3\alpha\sec^4\phi$, which decreases from sec α at the point of departure to $\cos^3\alpha$ at the vertex and then increases to $\cos^3\alpha\sec^4\omega$. Therefore, so long as α is not large, the value of β always lies between a maximum which is very little greater than unity and a minimum which is unity or very little less than unity.

It must be understood, in this connection, that the substitution of the constant β for the variable quantity $\frac{\cos^{n-2}\alpha}{\cos^{n-1}\phi}$ in any integral requires β to be given the mean value of $\frac{\cos^{n-2}\alpha}{\cos^{n-1}\phi}$ *when considered as a function of the variable whose differential remains under the integral sign.* For instance, β in (59) is the mean value of $\frac{\cos^{n-2}\alpha}{\cos^{n-1}\phi}$ regarded as a function of $\tan\phi$, while in (61) β is the mean value of $\frac{\cos^{n-2}\alpha}{\cos^{n-1}\phi}$ regarded as a function of x. Consequently β has a different value in each of the ballistic formulæ (67), just, for example, as the mean value of the ordinate of a semicircle has different values according as the ordinates are

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equally spaced along the diameter (y a function of x) or along the circumference (y a function of s) or still otherwise taken.

It happens that for the values 2, 3, and 5 of n the expression $\frac{\cos^{n-2}\alpha}{\cos^{n-1}\phi}\, d(\tan\phi)$ is directly integrable; also that $\frac{\cos^{n-2}\alpha}{\cos^{n-1}\phi}\, dx$ is integrable for $n = 2$ and approximately so for $n = 3$ and $n = 5$. So the exact value of β as a function of $\tan\phi$, and its exact, or very nearly exact, value as a function of x, can be calculated for any given trajectory over which the resistance is proportional to either the square, the cube, or the fifth power of the velocity.

The following table gives the calculated values of β for what are taken to be about the average trajectories of modern high-power guns with angles of departure of 5°, 10°, 15° and 20°. The values of $\sqrt{\sec\alpha}$ are given for comparison with β.

For a trajectory in which	β as a function of $\tan\phi$			β as a function of x			$\sqrt{\sec\alpha}$
	For n=2	For n=3	For n=5	For n=2	For n=3	For n=5	
$\alpha = 5°$ and $\omega = 8°$	1.0025	1.0012	0.9986	1.0020	1.0006	0.9975	1.0019
$\alpha = 10°$ and $\omega = 17°$	1.0117	1.0080	1.0012	1.0073	1.0047	0.9945	1.0077
$\alpha = 15°$ and $\omega = 26°$	1.0289	1.0235	1.0156	1.0169	1.0142	0.9960	1.0175
$\alpha = 20°$ and $\omega = 35°$	1.0578	1.0550	1.0605	1.0309	1.0337	1.0133	1.0316

Actually the value of n very rarely remains the same throughout any trajectory, usually passing through the successive values 1.7, 2 and 3 to 5; when the angle of departure exceeds 4° or 5° passing through the value 5 to 3 again; and in the case of still higher trajectories passing through the value 3 and reaching the value 2 a second time. Consequently, the greater the value of α, the greater the part of the trajectory over which n has the values 3 and 5.

If we suppose n to have the mean value 3, the table shows that β as a function of x may be taken as unity with less than 1% of error for trajectories in which α does not exceed about 13°, and that until α exceeds 10° the error caused by giving the value unity to β as a function of $\tan\phi$ is also less than 1%; therefore, considering that there are many other unavoidable causes of error and inaccuracy in the solution of ballistic problems, we are fully justified in assuming $\beta = 1$ in all Siacci's formulæ so long as we are only dealing with direct fire, and it is customary to make this assumption. Not until α exceeds about 15° is it desirable

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to give β a value other than unity, and then the approximate value $\beta = \sqrt{\sec \alpha}$ will be found to give good results.

We might, of course, divide any trajectory into such short arcs that the value of ϕ would change but slightly from one end to the other of each, in which case, taking α as the inclination at the beginning of an arc, β would differ from $\sec \alpha$ less and less the shorter the arc, so that the error could be made as small as we wished. Such a method, however, would be laborious and would not have the special advantage of Siacci's method as commonly used, namely, that of covering the whole trajectory by a single operation.

71. **The Correction for Altitude.**—Thus far we have supposed the density of the air to be constant over the whole path of the projectile, whereas it is evident that in the upper parts of a trajectory the air must be considerably less dense than at the gun's level. To take account of this a factor f, called the *altitude factor,* which is the ratio of the density of the air at the gun to the mean density of the air through which the projectile travels, is introduced into the numerator of the ballistic coefficient. The mean height of the projectile during its flight is taken to be two-thirds the height of the vertex of the trajectory, which would be exact if the trajectory were a parabola and is therefore practically correct, and the mean density of the air is taken to be the density at that height $\left(\frac{2y_0}{3}\right)$. Correction for altitude is a needless refinement for trajectories for which the time of flight does not exceed about 12 seconds, and the value of f for any height (h) will be found in Table III appended, h being taken as $\frac{2}{3}\, y_0$ and y_0 calculated by the formula (34) $y_0 = \frac{gT^2}{8}$.

72. **The Coefficient of Reduction** is the product $\frac{f}{\beta\delta}$ by which the ballistic coefficient $C = \frac{w}{id^2}$ is to be multiplied to obtain the reduced ballistic coefficient C', which has hitherto been written $C' = \frac{w}{\delta i d^2}$, but the complete value of which is now seen to be $\frac{fw}{\beta\delta i d^2}$. Hereafter we will write the ballistic formulæ with this understanding, replacing $\frac{C'}{\beta}$ in (67) by C', and giving f, β, δ and i values other than unity when the circumstances require it.

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EXAMPLES.—VI

(1) The 4000-yard trajectory of a 3-inch gun (I. V. = 2800 f. s.) has the following elements: $\alpha = 3^\circ\ 10'$; $\omega = 6^\circ\ 0'$; v_0 (velocity at vertex) = 1600 f. s.; v_ω = 1050 f. s. What are the pseudo velocities at the three points where the velocities are given?

$Z = 2800$; $z_0 = 1602$; $z_\omega = 1046$.

(2) In the 4000-yard trajectory of a 6-inch gun (I. V. = 2400 f. s.) $\alpha = 2^\circ\ 52'$; $\omega = 4^\circ\ 09'$; $v_0 = 1780$ f. s., and $v_\omega = 1377$ f. s. What are the pseudo velocities at point of departure, vertex and point of fall?

2400; 1782; 1375.

(3) In the 8000-yard trajectory of a 12-inch gun (I. V. = 2250 f. s.) $\alpha = 6^\circ\ 23'$; $\omega = 8^\circ\ 58'$; $v_0 = 1690$ f. s., and $v_\omega = 1347$ f. s. What are the pseudo velocities at point of departure, vertex and point of fall?

2250; 1701; 1339.

(4) In the 10,000-yard trajectory of an 8-inch gun (I. V. = 2300 f. s.) $\alpha = 11^\circ\ 00'$; $\omega = 18^\circ\ 40'$; $v_0 = 1340$ f. s., and $v_\omega = 1036$ f. s. What are the pseudo velocities at point of departure, vertex and point of fall?

2300; 1365; 1000.

(5) In the trajectory of example (1) what is the pseudo velocity at the point whose abscissa is 2000 yards, and what is the time of flight to that point (w = 13 lbs.)?

1671; 2.791 sec.

(6) In the trajectory of example (2) what is the pseudo velocity at the point whose abscissa is 2000 yards, and what is the time of flight to that point (w = 100 lbs.)?

1829; 2.867 sec.

(7) In the 4000-yard trajectory of a 5-inch gun (w = 50 lbs.), V = 2550 f. s., $\alpha = 3^\circ\ 01'$, what is the pseudo velocity at the point whose abscissa is 2200 yards? What is the value of ϕ at the same point?

$z = 1683$; $\phi = 7'\ 10''$.

(8) In the 6000-yard trajectory of a 6-inch gun (w = 100 lbs.), V = 2300 f. s., $\alpha = 5^\circ\ 53'$, what is the pseudo velocity after 1000

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and after 2000 yards horizontal travel? What are the ordinates of the trajectory at those distances?

$z = 2010;\ 1747.$
$y = 279$ ft; 485 ft.

(9) In the 3000-yard trajectory of a 3-inch gun ($w = 13$ lbs.), $V = 2800$ f. s., and $\alpha = 1° 53'$, what are the pseudo velocities after 1, 2, 3 and 4 seconds' flight?

2276; 1898; 1618; 1411.

(10) With the data of example (9) determine the horizontal distances traveled in the times stated.

840; 1533; 2118; 2619 yds.

(11) Determine the reduced ballistic coefficient for a 10-inch, 500-pound projectile of standard form if the temperature and barometric height at the gun are 84° and 29.12 and the time of flight is 16 seconds. From the given time of flight, the initial velocity being 2000 f. s., find the approximate values of the pseudo velocity at the point of fall and the range?

$C' = 5.432$; $z_\omega = 1151$; $X = 7929$ yds.

(12) Determine the reduced ballistic coefficient for an 8-inch 250-pound projectile of standard form, the temperature being 46°, the barometer 30.11, and the time of flight 22 seconds. If the initial velocity was 2300 f. s. and the angle of departure was about 11°, find z_ω from the given value of the time of flight and then find the range.

$C' = 3.858$; $z_\omega = 974$; $X = 9952$ yds.

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CHAPTER VII

BALLISTIC PROBLEMS.

73. Case I.—*Given the initial velocity* (V) *and the range* (X), *to find the angle of departure* (α), *the angle of fall* (ω), *the time of flight* (T), *and the final velocity* (v_ω).

Let z_ω be the pseudo velocity at the point of fall where $y = 0$, $x = X$, $t = T$, $v = v_\omega$, and $\phi = -\omega$. Then, putting $y = 0$ in the second of (67), and $\phi = -\omega$ in the third, we have

$$\tan\alpha = \frac{C'}{2\cos^2\alpha}\left[\frac{A_{z_\omega} - A_v}{S_{z_\omega} - S_v} - I_v\right] \text{ and } \tan\omega = \frac{C'}{2\cos^2\alpha}[I_{z_\omega} - I_v] - \tan\alpha,$$

from which, together with the other equations of (67), we derive the following formulæ for this case:

$$\left.\begin{aligned} S_{z_\omega} &= S_v + \frac{X}{C'} \\ \sin 2\alpha &= C'\left[\frac{A_{z_\omega} - A_v}{S_{z_\omega} - S_v} - I_v\right] \\ \tan\omega &= \frac{C'\sec^2\alpha}{2}\left[I_{z_\omega} - \frac{A_{z_\omega} - A_v}{S_{z_\omega} - S_v}\right] \\ T &= C'\sec\alpha\,[T_{z_\omega} - T_v] \\ v_\omega &= z_\omega\cos\alpha\sec\omega \end{aligned}\right\} \quad (68)$$

74. Examples Illustrating Case I.—*Example* (1). *An 8-inch projectile, weight 250 lbs., has an initial velocity of 2300 f. s. Find the angle of departure for a range of 3000 yards; also the angle of fall, the time of flight, and the final velocity at that range.*

Here, as always when not otherwise stated, the projectile must be taken to be of standard form ($i = 1$) and the air to be of standard density ($\delta = 1$). Then, taking $\beta = 1$ and $f = 1$, C' and $\frac{X}{C'}$ are found as follows:

$w = 250$	log	2.39794
$d^2 = 64$	"	1.80618
C'	"	.59176
$X = 9000$	"	3.95424
$\frac{X}{C'} = 2304.0$	"	3.36248

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Next, from Table I,

$V = 2300 \quad S_v = 3987.6 \quad A_v = 178.78 \quad I_v = .06297 \quad T_v = 1.388$

$\frac{X}{C'} = 2304.0$

$z_\omega = 1714 \ldots S_{z_\omega} = 6291.6 \quad A_{z_\omega} = 363.25 \quad I_{z_\omega} = .10093 \quad T_{z_\omega} = 2.550$

$*\Delta A = 184.47 \qquad *\Delta T = 1.162$

Now, with the foregoing data, we find a, as follows:

$\Delta A = 184.47$	log	2.26593
$\Delta S = 2304.0$	"	3.36248
$\frac{\Delta A}{\Delta S} = .08007$	"	8.90345
$I_v = .06297$		
$\frac{\Delta A}{\Delta S} - I_v = .01710$	"	8.23300
C'	"	.59176
$2a = 3° 49' 48''$	l. sin	8.82476
$a = 1° 55'.$		

Next, to find ω, we proceed as follows:

$I_{z_\omega} = .10093$		
$\frac{\Delta A}{\Delta S} = .08007$		
$I_{z_\omega} - \frac{\Delta A}{\Delta S} = .02086$	log	8.31931
C'	"	.59176
$a = 1° 55'$	2 l. sec	.00048
		8.91155
2	log	.30103
$\omega = 2° 20'$	l. tan	8.61052

Finally, to find T and v_ω

$\omega = 2° 20'$	l. sec	.00036		
$a = 1° 55'$	l. sec	.00024	l. cos	9.99976
C'	log	.59176		

*For brevity, the difference of two altitude functions is denoted by ΔA, the difference of two time functions by ΔT, &c. Note also that $\Delta S = S_{z_\omega} - S_v = \frac{X}{C'}$.

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$z_\omega = 1714$ ..... log 3.23401
$\Delta T = 1.162$ ..... " .06521
$T = 4.54$ ..... " .65721
* $v_\omega = 1714.5$ ..... " 3.23413

Thus we have as answers to the problem, $\alpha = 1° 55'$; $\omega = 2° 20'$; $T = 4.54$ sec.; and $v_\omega = 1714$ f. s.

75. *Example (2). A 6-inch gun with w = 100 pounds, and I. V. = 2900 f. s., gave a range of 8628 yards, the temperature being 70° and the barometer 30.05. Find the angle of departure, the angle of fall, the time of flight, and the striking velocity.*

Here, as it will probably be necessary to correct for altitude, we first find the time of flight, in order to determine f.

$w = 100$ ..... log 2.00000
$d^2 = 36$ ..... " 1.55630
.44370
$\delta = .995$ ..... " .99782
C' ..... " .44588
$X = 25884$ ..... " 4.41303
$\frac{X}{C'} = 9271.5$ 3.96715

$V = 2900$ .. $S_V = 2019.4$ ..... $T_V = .625$
$Z_\omega = 1000.2$. $S_{z_\omega} = 11290.9$ ..... $T_{z_\omega} = 6.551$
$\Delta T = 5.926$ ... log .77276
C' ..... " .44588
$T = 16.54$ ... " 1.21864

With this approximate value of T we compute the height of vertex $y_0 = \frac{gT^2}{8} = 1102$ feet; take $f = 1.019$ from Table III corresponding to $h = \frac{2}{3}y_0 = 735$; correct C' as follows; and then proceed as in example (1).

C' (first value) ..... log .44588
$f = 1.019$ ..... " .00817
C' (corrected) ..... " .45405
$X = 25884$ ..... " 4.41303
$\frac{X}{C'} = 9098.8$ 3.95898

* As shown by the result, it is unnecessary to compute the value of v_ω in such flat trajectories; it may be taken to be the same as z_ω.

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$V = 2900$ $S_v = 2019.4$ $A_v = 75.09$ $I_v = .04388$ $T_v = 0.625$

$\frac{X}{C'} = 9098.8$

$z_\omega = 1011.4$ $S_{z_\omega} = 11118.2$ $A_{z_\omega} = 1250.13$ $I_{z_\omega} = .30094$ $T_{z_\omega} = 6.379$

$\Delta A = 1175.04$ $\Delta T = 5.754$

$\Delta A = 1175.04$ .... log 3.07006

$\Delta S = 9098.8$ .... " 3.95898 $I_{z_\omega} = .30094$

$\frac{\Delta A}{\Delta S} = .12915$ .... " 9.11108 ... $\frac{\Delta A}{\Delta S} = .12915$

$I_v = .04388$ $I_{z_\omega} - \frac{\Delta A}{\Delta S} = .17179$ log 9.23500

$\frac{\Delta A}{\Delta S} - I_v = .08527$ .... " 8.93080 C' ..... " .45405

$\alpha = 7° - 01' \; 2$ l. sec .00652

C' ..... " .45405 9.69557

$2\alpha = 14° \; 02' \; 20''$ l. sin 9.38485 2 ..... log .30103

$\alpha = 7° \; 01' \; 10''$ $\omega = 13° \; 56'$ l. tan 9.39454

$\omega = 13° \; 56'$ ..... l. sec. .01297

$\alpha = 7° \; 01'$ ..... l. sec .00326 .. l. cos. 9.99674

C' ..... log .45405

$z_\omega = 1011.4$ ..... log. 3.00492

$\Delta T = 5.754$ ..... log .75997

$T = 16.49$ ..... " 1.21728

$v_\omega = 1034.3$ ..... " 3.01463

Thus we have as answers to the problem, $\alpha = 7° \; 01'$; $\omega = 13° \; 56'$; $T = 16.49$ sec.; and $v_\omega = 1034$ f. s. The actual measured angle of departure was $7° \; 02' \; 26''$.

76. Case II.—*Given the initial velocity (V) and the angle of departure (α), to find the coordinates of the vertex (x_0 y_0) and the corresponding time and velocity.*

Let z_0 be the pseudo velocity at the vertex, where $\phi = 0$, $t = t_0$, $v = v_0$, $x = x_0$ and $y = y_0$. Then, putting $\phi = 0$ in the third of (67), we have $\tan \alpha = \frac{C'}{2 \cos^2 \alpha} [I_{z_0} - I_v]$, from which, together

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with the other equations of (67), we derive the following formulæ for this case:

$$\left.\begin{aligned} I_{z_0} &= I_V + \frac{\sin 2\alpha}{C'} \\ x_0 &= C'\,[S_{z_0} - S_V] \\ y_0 &= x_0 \tan\alpha - \frac{C'\sec^2\alpha}{2}\,x_0\left[\frac{A_{z_0} - A_V}{S_{z_0} - S_V} - I_V\right] \\ t_0 &= C'\sec\alpha\,[T_{z_0} - T_V] \\ v_0 &= z_0\cos\alpha \end{aligned}\right\} \quad (69)$$

77. **Examples Illustrating Case II.**—*Example* (1). *The U. S. magazine rifle, caliber .30, gives 1000 yards' range with an angle of departure of 1° 36′. Determine the coordinates of the vertex and the corresponding time and velocity.*

Here we have w = 220 grains = .031429 pounds, d = .3, V = 2000 f. s., and X = 3000 feet.

w = .031429 . . . . log 8.49733
d^2 = .09 . . . . " 8.95424
C′ . . . . . " 9.54309
2α = 3° 12′ . . . . l. sin 8.74680
$\frac{\sin 2\alpha}{C'}$ = .15985 . . . . . log 9.20371

I_V = .07864 S_V = 5106.1 A_V = 257.57 T_V = 1.909
I_{z_0} = .23849
z_0 = 1094.3 S_{z_0} = 10048.7 A_{z_0} = 962.37 T_{z_0} = 5.360
ΔS = 4942.6 ΔA = 704.80 ΔT = 3.451

ΔA = 704.8 . . . . . log 2.84807
ΔS = 4942.6 . . log 3.69395 " 3.69395

C′ . . . . . " 9.54309 " 9.15412 $\frac{\Delta A}{\Delta S}$ = .14260
x_0 = 1726.0 . . " 3.23704 I_V = .07864
α = 1° 36′ l. tan 8.44611

$x_0 \tan\alpha$ = 48.21 . . 1.68315 $\frac{\Delta A}{\Delta S} - I_V$ = .06396 log 8.80591
19.29 C′ . . . . . " 9.54309
y_0 = 28.92 x_0 . . . . . " 3.23704
α . . . . 2 l. sec .00034
log 1.58638
2 . . . . . " .30103
19.29 . . . . " 1.28535

ΔT = 3.451 . . log .53794
C′ . . . . . " 9.54309
α . . . . . l. sec .00017
t_0 = 1.206 . . log .08120
$v_0 = z_0$ = 1094

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Thus we have for the coordinates of the vertex $x_0 = 1726$ feet, $y_0 = 28.92$ feet; for the time of flight to the vertex 1.206 seconds, and for the velocity at the vertex 1094 f. s. The official description of this rifle gives 1712 feet and 28.67 feet as the coordinates of the vertex determined by experiment.

78. *Example (2). Determine the coordinates of the vertex and the corresponding time and velocity in the case of the trajectory of example (2) under Case I; that is, given $V = 2900$ f. s., $\alpha = 7° 01' 10''$, and log $C' = .45405$.*

$2\alpha = 14° 02' 20''$ . . . . l. sin 9.38485
C' . . . . . log .45405

$\frac{\sin 2\alpha}{C'} = .08527$ 8.93080

$I_V = .04388$ . . . . . $S_V = 2019.4$ $A_V = 75.09$ $T_V = 0.625$
$I_{z_0} = .12915$
$z_0 = 1481.0$ $S_{z_0} = 7401.3$ $A_{z_0} = 490.24$ $T_{z_0} = 3.248$
$\Delta S = 5381.9$ $\Delta A = 415.15$ $\Delta T = 2.623$

$\Delta A = 415.15$ . . . . . log 2.61821
$\Delta S = 5381.9$ . . . . . log 3.73093 " 3.73093
C' . . . . . " .45405 " 8.88728

$\frac{\Delta A}{\Delta S} = .07714$

$x_0 = 15310$ . . . . . " 4.18498
$\alpha = 7° 01' 10''$ . . . l. tan 9.09035 $I_V = .04388$

$x_0 \tan\alpha = 1885.1$ . . . . . log 3.27533 $\frac{\Delta A}{\Delta S} - I_V = .03326$ log 8.52192
735.3 C' . . . . . " .45405
$y_0 = 1149.8$ x_0 . . . . . " 4.18498
α . . . . 2 l. sec .00652
3.16747
2 . . . . . log .30103
735.3 . . . 2.86644

$z_0 = 1481$ . . . . . log 3.17056
$\Delta T = 2.623$ . . . log .41880
C' . . . . . " .45405
α . . . . . l. sec .00326 . . l. cos 9.99674
$t_0 = 7.52$ . . . log .87611
$v_0 = 1470$ . . . . . log 3.16730

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Thus we have for the coordinates of the vertex $x_\theta = 15{,}310$ feet; $y_0 = 1150$ feet; for the time of flight to the vertex 7.52 seconds, and for the velocity at the vertex 1470 f. s. In **75**, where the height of the vertex was determined from the time of flight, for the purpose of correcting the ballistic coefficient for altitude, we found $y_0 = 1101$ feet, from which we derived $f = 1.019$; if we should take the more accurate value, $y_0 = 1150$ feet, just found, we should have $f = 1.020$, but the resulting difference in C′ would not materially affect our results.

79. The foregoing results, together with those found for the same data in **75**, enable us to compare the loss of velocity, the horizontal distance passed over, and the time taken in describing the ascending and the descending branches of this trajectory, for which $\alpha = 7° 01'$ and $V = 2900$ f. s. They are as follows:

	Loss of Velocity.	Horizontal Distance.	Elapsed Time.
Ascending branch	1430 f. s.	5103 yds.	7.52 sec.
Descending branch	436 f. s.	3525 yds.	8.97 sec.
Entire trajectory	1876 f. s.	8628 yds.	16.49 sec.

The considerable difference between the horizontal projections of the two branches and the relatively small difference between the times taken in describing them should be noted. The great difference in the rates of retardation in the two branches is also noteworthy, and a comparison of the corresponding losses of energy is still more striking. The kinetic energy of the projectile at the origin is 13,060,000 ft. lbs.; at the vertex it is 3,355,000 ft. lbs.; and at the point of fall it is only 1,660,000 ft. lbs., the resistance of the air having absorbed 11,400,000 ft. lbs. of enery during the flight. At the vertex the projectile has also a potential energy of 115,000 ft. lbs.

EXAMPLES.—VII

(CASE I)

(1) Given $d = 3''$, $w = 13$ lbs., $V = 2800$ f. s., and $X = 2000$ yds. Determine α, ω, T and v_ω.

1° 01′; 1° 26′; 2.79 sec.; 1672 f. s.

(2) Given $d = 5''$, $w = 50$ lbs., $V = 2550$ f. s. and $X = 3000$ yds. Determine α, ω, T and v_ω.

1° 55′; 2° 49′; 4.74 sec.; 1439 f. s.

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(3) Given d = 6″, w = 100 lbs., V = 2300 f. s., and X = 4000 yds. Determine α, ω, T and v_ω.

3° 07′; 4° 31′; 6.96 sec.; 1320 f. s.

(4) Given d = 8″, w = 250 lbs., V = 2300 f. s., and X = 4000 yds. Determine α, ω, T and v_ω.

2° 45′; 3° 35′; 6.38 sec.; 1553 f. s.

(5) Given d = 12″, w = 850 lbs., V = 2250 f. s., and X = 5000 yds. Determine α, ω, T and v_ω.

3° 25′; 4° 15′; 7.9 sec.; 1628 f. s.

(6) *A Krupp 28 cm. (11.024″) gun, with w = 760.4 lbs. and V = 1733 f. s., gave a mean range of 6780 feet for $\alpha = 2° 23'$, the weight of a cubic meter of the air being 1.245 kilos and that of standard air being 1.208 kilos. Compute the values of α and of v_ω.

2° 18′; 1496 f. s.

(7) † The gun of example (6), with $\delta = \frac{1.215}{1.208}$, gave a mean range of 20,364 feet for $\alpha = 8° 30'$. Compute the values of α and of z_ω, correcting C′ for altitude.

8° 34′; 1157 f. s.

(8) The 6-inch gun, w = 100 lbs., V = 2900 f. s., gave a measured range of 9700 yards for $\alpha = 9° 02' 10''$, the thermometer being at 70° and the barometer 30.05. Compute the values of α and z_ω, correcting C′ for altitude.

8° 53′; 950 f. s.

(9) The mean of three shots with the 12-inch gun, w = 850 lbs., gave a range of 11,556 yards with an I. V. of 2827 f. s., thermometer 59°, barometer 30.19. Compute the value of α, z_ω and T, correcting C′ for altitude. (The angle of elevation was 7° 02′.)

6° 55′; 1348 f. s.; 18.14 sec.

(10) ‡ The gun of example (6) gave a range of 33,622 feet for $\alpha = 17° 15'$, the value of δ being $\frac{1.229}{1.208}$. Compute the values of α and T, correcting C′ for altitude. (Mean of observed times of flight was 27.6 sec.)

17° 47′; 28.2 sec.

*There was a wind component of 19 f. s. directly against the flight.
†There was a wind component of 14 f. s. directly with the flight.
‡There was a wind component of 12 f. s. directly with the flight.

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(CASE II)

(11) The angle of departure for the .30 cal. small arm (w = 220 grains, V = 2000 f. s.) for 500 yds. range is 31′ 35″. Determine the coordinates of the vertex and the corresponding time and velocity. (Compare with approximate values, Chap. IV, 42.) 826 ft.; 4.18 ft.; 0.48 sec.; 1467 f. s.

(12) The angle of departure for the 6-inch gun (w = 100 lbs., V = 2900 f. s.) for 2625 yds. range was 1° 07′ 49″. Determine x_0, y_0, t_0 and v_0. (Compare with approximate values, Examples IV (2)).

1430 yds.; 44.9 ft.; 1.62 sec.; 2421 f. s.

(13) The angle of departure for the 6-inch gun (w = 100 lbs., V = 2900 f. s.) for a range of 5394 yds. was 3° 03′ 51″. Determine x_0, y_0, t_0 and v_0. (Compare with approximate values, Examples IV (4).)

3063 yds.; 279 ft.; 3.89 sec.; 1941 f. s.

(14) The Krupp 28 cm. (11.024″) gun, w = 760.4 lbs., V = 1733 f. s., gave a range of 33,622 feet for $\alpha = 17° 15'$. The value of δ was $\frac{1.229}{1.208}$. Determine x_0, y_0, v_0 and t_0, correcting C′ for altitude. (For whole trajectory see Ex. (10).)

18024 ft.; 3132 ft.; 1159 f. s.; 13.13 sec.

(15) The 12-inch gun, w = 850 lbs., when fired with 2827 f. s. I. V. at an angle of elevation of 7° 02′ gave a range of 11,556 yds., the value of δ being 1.023. Determine x_0, y_0, v_0 and t_0, correcting C′ for altitude. (For whole trajectory see Ex. (9).)

6570 yds.; 1379 ft.; 1867 f. s.; 8.63 sec.

(CASE I)

(16) The range table for a Krupp 40 cm. (15.75″) gun, w = 2028 lbs., V = 1805 f. s., gives $\alpha = 1° 00'$ for X = 1000 meters (3281 ft.). Compute the values of α and v_ω for this range.

57′ 46″; 1712 f. s.

(17) The same range table gives $\alpha = 3° 04'$ for X = 3000 meters (9842 ft.). Compute the values of α and v_ω.

3° 06′ 34″; 1541 f. s.

(18) The same range table gives $\alpha = 5° 34'$ for X = 5000 meters (16,404 ft.). Compute the value of α and v_ω.

5° 36′ 36″; 1386 f. s.

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(19) Given d = 3″, w = 13 lbs., V = 2800 f. s., and X = 2000 yds. Find α and v_ω. 1° 00′ 43″; 1671 f. s.

(20) Given d = 3″, w = 15 lbs., V = 2628 f. s., and X = 1883 yds. Find α and v_ω. 1° 00′ 42″; 1720 f. s.

(21) Given d = 5″, w = 60 lbs., V = 2900 f. s., and X = 3000 yds. Find α and v_ω. 1° 21′ 16″; 1836 f. s.

(22) Given d = 5″, w = 55 lbs., V = 2997 f. s., and X = 3095 yds. Find α and v_ω. 1° 21′ 42″; 1794 f. s.

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CHAPTER VIII.

BALLISTIC PROBLEMS—Continued.

80. Case III.—*Given the initial velocity* (V) *and the angle of departure* (α), *to find the range* (X), *the time of flight* (T), *the angle of fall* (ω), *and the final velocity* (v_ω).

Putting $y = 0$ in the second of the ballistic formulæ (67), we obtain, as in 73, $\sin 2\alpha = C' \left[\frac{A_{z_\omega} - A_v}{S_{z_\omega} - S_v} - I_v\right]$, from which, by trial and error, the value of the pseudo velocity at the point of fall (z_ω) may be determined. Then, by means of the other equations of (68), X, ω, T and v_ω are found. Noting that $I_v + \frac{\sin 2\alpha}{C'} = I_{z_0}$ (equations (69)), the formulæ for Case III are readily seen to be as follows:

$$\left.\begin{aligned} \frac{A_{z_\omega} - A_v}{S_{z_\omega} - S_v} = I_v + \frac{\sin 2\alpha}{C'} &= I_{z_0} \\ X &= C'(S_{z_\omega} - S_v) \\ \tan\omega &= \frac{C' \sec^2 \alpha}{2}(I_{z_\omega} - I_{z_0}) \\ T &= C' \sec\alpha\,(T_{z_\omega} - T_v) \\ v_\omega &= z_\omega \cos\alpha \sec\omega \end{aligned}\right\} \quad (70)$$

To find z_ω we assume two values for it and calculate the corresponding values of the first member of the equation $\frac{A_{z_\omega} - A_v}{S_{z_\omega} - S_v} = I_{z_0}$. Then, by comparing the results with the known value of the second member (I_{z_0}), we obtain the true value of z_ω on the assumption that variations of z_ω cause proportional variations of $\frac{A_{z_\omega} - A_v}{S_{z_\omega} - S_v}$. This latter assumption is only true when the two assumed values of z_ω are not very different from its true value, and to insure this we use the following method of selecting them:

81. *The approximate value of* z_ω. The time of flight to the vertex is $t_0 = C' \sec\alpha\ [T_{z_0} - T_v]$ and the whole time of flight

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is $T = C' \sec \alpha\ [T_{z_\omega} - T_V]$. If the latter were just double the former, we should have

$$C' \sec \alpha\ [T_{z_\omega} - T_V] = 2\ C' \sec \alpha\ [T_{z_0} - T_V]$$

$$T_{z_\omega} = 2\ T_{z_0} - T_V \qquad (71)$$

But, since T is really slightly greater than twice t_0, (71) would give too small a value for T_{z_ω}, and so, to get the two trial values of z_ω, proceed as follows:

Find z_0 by means of $I_{z_0} = I_V + \frac{\sin 2\alpha}{C'}$, and take T_{z_0} from the table; add 1.1 $(T_{z_0} - T_V)$ to T_{z_0} for a first approximate value of T_{z_ω} and add 1.2 $(T_{z_0} - T_V)$ to T_{z_0} for a second approximate value of T_{z_ω}; then take the two corresponding values of z_ω from the table, each to the nearest whole number, and use them as the trial values.

82. Examples Illustrating Case III.—*Example (1). The 4-inch gun, w = 33 pounds, V = 2200 f. s., is fired with* $\alpha = 3^\circ\ 04'$*; determine the range, time of flight, angle of fall, and striking velocity.*

$V = 2200$ $S_V = 4348.4$ $A_V = 202.32$ $I_V = .06755$ $T_V = 1.548$

$T_{z_0} = 3.019$

$\Delta T = 1.471$

$w = 33$ ... log 1.51851

$d^2 = 16$ ... " 1.20412

C' ..... " .31439

$2\alpha = 6^\circ\ 08'$ l. sin 9.02874

$\frac{\sin 2\alpha}{C'} = .05180$ log 8.71435

$I_V = .06755$

$I_{z_0} = .11935$ .... $z_0 = 1550$ (nearest tabular value)

$T_{z_0} = 3.019$	3.019
$1.1\Delta T = 1.618$	$1.2\Delta T = 1.765$
4.637	4.784
$z'_\omega = 1180$	$z''_\omega = 1160$

$S'_{z_\omega} = 9236.0$	$A'_{z_\omega} = 785.50$	$S''_{z_\omega} = 9404.5$	$A''_{z_\omega} = 819.50$
$S_V = 4348.4$	$A_V = 202.32$	$S_V = 4348.4$	$A_V = 202.32$
$\Delta' S = 4887.6$	$\Delta' A = 583.18$	$\Delta'' S = 5056.1$	$\Delta'' A = 617.18$

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$\Delta'A$ . . log 2.76580	$\Delta''A$ . . log 2.79042
$\Delta'S$ . . " 3.68909	$\Delta''S$ . . " 3.70382
" 9.07671	" 9.08660
$\frac{\Delta'A}{\Delta'S} = .11932$	$\frac{\Delta''A}{\Delta''S} = .12207$
$I_{z_0} = .11935$	.11932
3	275

$$z_\omega = 1180 - \frac{3}{275}(1180 - 1160) = 1179.8$$

The values of $\frac{A_{z_\omega} - A_V}{S_{z_\omega} - S_V}$, calculated with assumed values of z_ω respectively 1180 and 1160, being .11932 and .12207, while its true value is $I_{z_0} = .11935$, the true value of z_ω is found as above by proportion to be 1179.8.

Having thus determined z_ω, we proceed as follows:

$S_V = 4348.4$	$I_{z_0} = .11935$	$T_V = 1.548$
$z_\omega = 1179.8$ $S_{z_\omega} = 9237.6$	$I_{z_\omega} = .19793$	$T_{z_\omega} = 4.645$
$\Delta S = 4889.2$	$\Delta I = .07858$	$\Delta T = 3.097$
ΔS . . . . . log 3.68924	ΔI . . . . log 8.89531	ΔT . . . log .49094
C' . . . . . " .31439	C' . . . . " .31439	C' . . . " .31439
$X = 10084$ " 4.00363	$\sec^2\alpha$ . . . . " .00124	$\sec\alpha$ . . . " .00062
3361 yds.	" 9.21094	$T = 6.397$ " .80595
	2 . . . . " .30103	
z_ω . . . . . log 3.07184	$\omega = 4°\ 39'$ l. tan 8.90991	
$\cos\alpha$ . . . . " 9.99938		
$\sec\omega$ . . . . " .00143		
$v_\omega = 1182$ 3.07265		

Thus we have finally $X = 3361$ yds.; $T = 6.397$ sec.; $\omega = 4°\ 39'$, and $v_\omega = 1182$ f. s. (The measured range for this angle of departure, δ being unity, was 3325 yds., and the observed time of flight was 6.3 sec.)

83. *Example (2). A Krupp 28 cm. (11.024") gun, with $w = 760.4$ lbs. and $V = 1733$ f. s., was fired at an angle of elevation of 17° 15', the value of δ being $\frac{1.229}{1.208}$. Compute the range and time of flight.*

Here C' must be corrected for altitude, and, as the height of vertex of this trajectory has been already determined to be 3132

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feet (Examples VII, (14)), we will take f for $h = \frac{2}{3} \times 3132 =$ 2088 feet instead of recalculating it.

$w = 760.4$ . . . . . log 2.88104
$d = 11.024$ . . . . . 2 " 2.08468
$\frac{w}{d^2}$ . . . . . " .79636
$\delta = 1.0174$ . . . . . " .00748
" .78888
$f = 1.055$ . . . . . " .02325
C' . . . . . " .81213
$2\alpha = 34° 30'$ . . . . l. sin 9.75313
$\frac{\sin 2\alpha}{C'} = .08730$ . . . . . log 8.94100
$I_v = .09913$
$I_{z_0} = .18643$ . . . . . $z_0 = 1213$

$V = 1733.$
$S_v = 6207.1$
$A_v = 354.89$
$I_v = .09913$
$T_v = 2.500$, $T_{z_0} = 4.431$ } $\Delta T = 1.931$

$T_{z_0} = 4.431$ 4.431
$1.1\Delta T = 2.124$ $1.2\Delta T = 2.317$
6.555 6.748
$z'_\omega = 1000$ $z''_\omega = 988$

$S'_{z_\omega} = 11294.7$ $A'_{z_\omega} = 1304.05$ $S''_{z_\omega} = 11488.8$ $A''_{z_\omega} = 1365.82$
$S_v = 6207.1$ $A_v = 354.89$ $S_v = 6207.1$ $A_v = 354.89$
$\Delta'S = 5087.6$ $\Delta'A = 949.16$ $\Delta''S = 5281.7$ $\Delta''A = 1010.93$

$\Delta'A$ . . log 2.97734
$\Delta'S$ . . " 3.70651
" 9.27083
$\frac{\Delta'A}{\Delta'S} = .18657$
$I_{z_0} = .18643$
14

$\Delta''A$ . . log 3.00472
$\Delta''S$ . . " 3.72278
" 9.28194
$\frac{\Delta''A}{\Delta''S} = .19140$
.18657
483

$$z_\omega = 1000 + \frac{14}{483}(1000 - 988) = 1000.35$$

Then, having found z_ω,—

$S_{z_\omega} = 11289.4$ $T_{z_\omega} = 6.549$
$S_v = 6207.1$ $T_v = 2.500$
$\Delta S = 5082.3$ $\Delta T = 4.049$

ΔS . . . log 3.70606
C' . . . " .81213
$X = 32975$ . . . " 4.51819

ΔT . . . log .60735
C' . . . " .81213
sec α . . . " .01999
$T = 27.51$ . . . " 1.43947

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The mean result of five rounds fired at this elevation at Krupp's Proving Grounds at Meppen was a range of 33,622 feet and a time of flight of 27.6 seconds, there being at the time of firing a wind of about 12 f. s. velocity directly with the flight of the projectiles.

84. Case IV.—*Given the range (X) and the angle of departure (a), to calculate the initial velocity (V).*

In this case we have to find two unknown quantities, V and z_ω so as to satisfy the first two equations of (68), which, after substituting in the second equation the value of $S_{z_\omega} - S_V$ from the first, may be written as follows:

$$\left.\begin{aligned} S_{z_\omega} &= S_V + \frac{X}{C'} \\ A_{z_\omega} - A_V - \frac{X}{C'} I_V &= \frac{X}{C'^2} \sin 2\alpha \end{aligned}\right\} \quad (72)$$

We assume two values of V, such that the true value of V will probably lie between them, and, by means of the first of (72), determine the two corresponding values of z_ω. Then, substituting in the second of (72), we compare the resulting values of the first member of that equation with its true value as given by the second member and thus by proportion arrive at the true value of V.

85. Example Illustrating Case IV.—*The 5-inch gun, w = 60 pounds, gave a range of 4670 yards with an angle of departure of 3°, the temperature being 40° F. and the barometric height 29.96. Determine the initial velocity, which was between 2700 f. s. and 2800 f. s.*

$\frac{w}{d^2} = 2.4$ .... log .38021
$\delta = 1.055$.... " .02325
C' ..... " .35696

$V_1 = 2700$ { $S_V = 2644.8$, $A_V = 104.09$, $I_V = .04902$ }
$V_2 = 2800$ { $S_V = 2329.1$, $A_V = 89.05$, $I_V = .04633$ }

$2\alpha = 6°$ ..... l. sin 9.01923
$X = 14010$ ..... log 4.14644
C' ..... " .35696

$\frac{X}{C'}$ =	6158.6	6158.6	" 3.78948	3.78948	3.78948 3.78948
				2.80871	
S_V =	2644.8	2329.1	log C'	.35696	log I_V 8.69037 8.66586
S_{z_ω} =	8803.4	8487.7	log	2.45175	log 2.47985 2.45534

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$z_\omega = 1238.4$	1286.6	$\frac{X \sin 2\alpha}{C'^2} = 282.98$	$\frac{X}{C'} I_V = 301.89$ 285.33
$A_{z_\omega} = 704.13$	649.65		
$A_V = 104.09$	89.05		
$\Delta A = 600.04$	560.60		
301.89	285.33	$V = 2800 - \frac{7.71}{22.88}(2800 - 2700) = 2766$ f. s.	
298.15	275.27		
275.27	282.98		
22.88	7.71		

The values of $A_{z_\omega} - A_V - \frac{X}{C'} I_V$ for assumed values of V, respectively 2700 and 2800, being 298.15 and 275.27, while its true value is $\frac{X}{C'^2} \sin 2\alpha = 282.98$, we find the true value of V by proportion as above.

Since any change in X will cause a larger proportional change in the calculated value of V, it is evident that this is a very crude method of determining V: only when X is the mean of a large number of very accurately observed results can the calculated value of V be taken to be a close approximation to its true mean value.

86. Case V.—*Given the initial velocity (V) and the coordinates (x, y) of a point to be hit, to find the angle of departure* (α).

Since $\frac{y}{x}$ is the tangent of the angle of position (p), the first two of the ballistic formulæ (67) may be written

$$S_z = \frac{x}{C'} + S_V$$

$$\tan p = \tan \alpha - \frac{C'}{2 \cos^2 \alpha}\left[\frac{A_z - A_V}{S_z - S_V} - I_V\right]$$

in each of which z is the pseudo velocity at the point whose abscissa is x, and, as shown by the first equation, is entirely independent of α, y, or anything else excepting V and C'.

But, putting $y = 0$ in the second of (67), and calling α_x the angle of departure for the horizontal range x, we have

$$\sin 2\alpha_x = C'\left[\frac{A_z - A_V}{S_z - S_V} - I_V\right]$$

in which again z is the pseudo velocity at the horizontal distance x from the origin.

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Consequently, combining the last two equations,

$$\tan p = \tan a - \frac{\sin 2a_x}{2\cos^2 a} = \frac{\sin 2a - \sin 2a_x}{2\cos^2 a}$$

from which, putting $2\cos^2 a = 1 + \cos 2a$ and $\tan p = \frac{\sin p}{\cos p}$, we derive

$$\sin(2a - p) = \cos p(\sin 2a_x + \tan p).$$

Thus we have the following formulæ for the solution of Case V:

$$\left.\begin{aligned} S_x &= \frac{x}{C'} + S_V \\ \sin 2a_x &= C'\left[\frac{A_x - A_V}{S_x - S_V} - I_V\right] \\ \tan p &= \frac{y}{x} \\ \sin(2a - p) &= \cos p(\sin 2a_x + \tan p) \end{aligned}\right\} \quad (73)$$

87. Example Illustrating Case V.—*What must the angle of departure be for a 6-inch 100-pound projectile, $V = 2000$ f. s., in order that it may hit a point whose coordinates are $x = 3500$ yards, $y = 500$ yards?*

$w = 100$ . . . . log 2.00000 $V = 2000$ . . $A_V = 257.57$
$d^2 = 36$ . . . . " 1.55630 $z = 1226.4$ . . $A_x = 719.04$
C' . . . . . " 44370 $\Delta A = 461.47$ . . log 2.66414
$x = 10500$ . . " 4.02119 $\Delta S = 3780.0$ . . " 3.57749
$\frac{x}{C'} = 3780.0$ . . " 3.57749 $\frac{\Delta A}{\Delta S} = .12208$ . . " 9.08665
$S_V = 5106.1$ $I_V = .07864$
$S_x = 8886.1$ . . $z = 1226.4$ .04344 . . " 8.63789
C' . . . . . " .44370
$y = 1500$ . . . log 3.17609 $2a_x = 6°55'51''$ l. sin 9.08159
$x = 10500$ . . " 4.02119 $\sin 2a_x = .12067$
$p = 8°07'48''$ l. tan 9.15490 . . . . . $\tan p = .14286$ . . l. cos 9.99562
.26353 . . log 9.42083
$2a - p = 15°\ 07'\ 23''$ . . l. sin 9.41645
$2a = 23°\ 15'\ 11''$
$a = 11°\ 37'\ 36''$

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The necessary angle of departure is thus found to be 11° 37′ 36″, and, the angle of position being 8° 07′ 48″, the necessary angle of projection is $\theta' = a - p = 3° 29' 48''$. The fact that the angle of departure for the horizontal range 3500 yards is $a = 3° 27' 56''$ shows how nearly accurate the assumption of the rigidity of the trajectory would be in this case. (The range on the incline is 3500 sec p = 3535 yds.)

88. In all problems where the height of the point aimed at is such that the rigidity of the trajectory cannot safely be taken for granted, the ballistic coefficient should be corrected for altitude,* and an examination of figure 10 will show that the mean height of the trajectory from gun to target is less or greater than

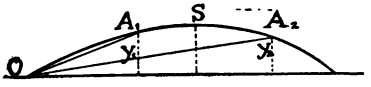

Fig. 10.

two-thirds the height of the target according as the latter is nearer to or further from the gun than the vertex of the complete trajectory. Thus, if the target is at A_1, the mean height of the arc OA_1 is less than $\frac{2}{3}y_1$ and approaches $\frac{y_1}{2}$ more and more as A_1 is nearer and nearer O: while if the target is at A_2 the mean height of the arc $O A_2$ is greater than $\frac{2}{3}y_2$, being about equal to y_2 when the abscissa of A_2 is $\frac{3}{4}$ X.

If, therefore, all possible refinements are to be introduced into the calculations, the relative positions of target and vertex of trajectory must be determined before fixing the value of f. Generally speaking, however, it will be sufficiently accurate to give f the value which corresponds to $h = \frac{2}{3}y_1$.

*The correction for altitude was purposely omitted in the solution of the example in **87** in order that the angle of departure for the horizontal range 3500 yds. might be determined at the same time as that for the inclined range whose horizontal projection is 3500 yds. If the value f = 1.026, corresponding to $h = \frac{2}{3} \cdot 500$ yds. = 1000 ft., had been used, a would have come out about 2′ less than it did.

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EXAMPLES.—VIII.

(CASE III)

(1) The 5-inch gun, w = 50 lbs., was fired with 2840 f. s. I. V. with an angle of elevation of 2°, thermometer 40°, barometer 29.96; determine the range and time of flight (measured range 3540 yds.).

3521 yds.; 5.34 sec.

(2) The 6-inch gun, w = 100 lbs., was fired with 2900 f. s. I. V. with an angle of elevation of 1° 07′ 49″, thermometer 70°, barometer 30.05; find the range and time of flight (measured range 2625 yds.).

2707 yds.; 3.34 sec.

(3) The 6-inch gun of example (2), under the same conditions, except with an elevation of 5° 02′ 51″, gave a measured range of 7298 yards; calculate the range and time of flight (correct for altitude, approximate time of flight being 12.5 sec.).*

7274 yds.; 12.66 sec.

(4) The 12-inch gun, w = 850 lbs., was fired with 2829 f. s. I. V. (mean of three rounds) with an angle of elevation of 7° 32′, thermometer 59°, barometer 30.19; find the range and time of flight (correct for altitude, approximate time of flight 19 sec.) (mean of the measured ranges 11,900 yds.).

12,175 yds.; 19.55 sec.

(5) The gun of example (4) was fired with the same angle of elevation (7° 32′), thermometer 85°, barometer 30.32, mean measured initial velocity 2794 f. s.; calculate the range and time of flight (mean measured range 11,905 yds., approximate time of flight 19 sec.).

12,223 yds.; 19.50 sec.

(6) The Krupp 28 cm. (11.024″) gun, w = 760.4, V = 1733 f. s., was fired with 2° 23′ elevation, $\delta = \frac{1.245}{1.208}$; find the range and time of flight (the mean measured range was 2260 yds., there being a wind velocity of 19 f. s. against the flight).

2339 yds.; 4.38 sec.

* If the time of flight were not given it would be necessary to calculate the height of vertex in order to determine the altitude correction.

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(7) The gun of example (6) was fired with 4° 54′ elevation, $\delta = \frac{1.209}{1.208}$: find the range and time of flight (the mean measured range was 4425 yds., there being a wind of 31 f. s. with the flight).

4379 yds.; 8.74 sec.

(8) The gun of examples (6) and (7) was fired with 8° 30′ elevation, $\theta = \frac{1.215}{1.208}$: find the range and time of flight (approximate time of flight for altitude correction 14.5 sec). (The mean measured range was 6788 yds., there being a wind of 13 f. s. with the flight.)

6747 yds.; 14.66 sec.

(9) Given d = 4″, w = 33 lbs., V = 2900 f. s., $\alpha = 7° 19′ 15″$, and $\delta = 0.98$; find X (use f = 1.017).

7607 yds.

(10) Given d = 4″, w = 33 lbs., V = 2900 f. s., $\alpha = 13° 16′ 43″$. and $\delta = 0.98$; find X (use f = 1.044).

10,094 yds.

(11) Given d = 6″, w = 100 lbs., V = 2400 f. s., $\alpha = 2° 50′ 27″$; find X and T.

3981 yds.; 6.63 sec.

(12) Given d = 6″, w = 100 lbs., V = 2400 f. s., $\alpha = 4° 31′ 09″$; find X and T.

5375 yds.; 9.98 sec.

(CASE IV)

(13) The 6-inch gun, w = 100 lbs., gave a range of 3995 yds. with $\alpha = 3° 04′ 15″$; find the initial velocity, which is known to be between 2300 and 2500 f. s.

2321 f. s.

(14) An 8-inch gun, w = 250 lbs., gave 6000 yds. range with $\alpha = 8° 35′$, temperature 83°, barometer 30 inches; find V, which is between 1600 and 1800 f. s.

1731 f. s.

(15) A 5-inch gun, w = 50 lbs., gave a range of 3540 yds. with $\alpha = 2°$, thermometer 40°, barometer 29.96. Find V, which is between 2800 and 2900 f. s.

2851 f. s.

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(16) Given $d = 6''$, $w = 100$ lbs., $\alpha = 1° 05'$, and $X = 2616$ yds.; find V, which is between 2860 and 2940 f. s.

2902 f. s.

(17) Given $d = 13''$, $w = 1100$ lbs., $\alpha = 1° 08'$ and $X = 2000$ yds.; find V, which is between 2260 and 2340 f. s.

2282 f. s.

(CASE V)

(18) The Signal Station at Gibraltar is 1270 ft. above the sea; what should the angle of projection be to hit it with a 13-inch gun, $w = 1100$ lbs, $V = 2300$ f. s., from a ship 3000 yards from its base ($f = 1.019$)?

$p = 8° 01' 55''$; $\theta = 1° 46' 07''$.

(19) Supposing the 13-inch gun of example (18) to be fired from the Signal Station at the ship, what should the angle of projection be ($f = 1.019$ as before, since in each case we assume $\delta = 1$ at the sea level)?

$p = -8° 01' 55''$; $\theta' = 1° 45' 10''$.

(20) It is desired to attack an observation balloon with a 3-inch field gun, $w = 15$ lbs., $V = 1750$ f. s.; what should the angle of departure be, the height of the balloon being 1500 feet and its horizontal distance from the gun 3300 yards ($f = 1.026$)?

$p = 8° 36' 56''$; $\theta' = 5° 10' 50''$.

(21) The Colt gun, caliber .30, $w = .03143$ lbs., $V = 2000$ f. s., fires from a top 90 feet high at a torpedo boat distant 400 yards; compare the proper angle of projection with that for a horizontal range of 400 yds.

$\theta' = 22' 52''$; $\alpha_x = 22' 53''$.

(22) A fort 900 feet above the sea is to be attacked with the 8-inch gun ($w = 250$ lbs., $V = 2300$ f. s.) from a ship distant 4500 yds.; what should the angle of departure and projection be ($f = 1.016$)?

$p = 3° 48' 51''$; $\theta' = 3° 13' 06''$.

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CHAPTER IX.

DRIFT.—THE THEORY OF SIGHTS.

89. Drift.—Experience shows that the projectiles of any rifled guns, when fired in a still atmosphere, deviate from the plane of fire (a vertical plane through the axis of the gun) to an extent approximately proportional to the square of the time of flight and in the direction towards which the upper surface of the projectile moves in its rotation. This deviation is called "drift," and for all United States guns is to the right, since they are so rifled that their projectiles, viewed from the rear, turn with the hands of a watch.

Since the drift increases more rapidly than the range, the horizontal projection of the trajectory is really a curve, convex to the horizontal trace of the plane of fire. Nevertheless, though the absolute value of the drift, especially at long ranges, is too great to be neglected in the practical use of guns, its relative value when compared with the range is always small enough to justify the assumption hitherto made that the trajectory is a plane curve. In other words, the actual trajectory differs so little from its projection on the plane of fire that no appreciable error results from regarding them as coincident in taking account of the effect of the resistance of the air upon the range, velocity, time of flight, etc.

90. The cause of drift is that soon after the projectile leaves the gun the line of action of the air resistance ceases to coincide with the axis of the projectile, on account of the curvature of the trajectory, and, meeting that axis obliquely between the point of the projectile and its center of gravity, tends to raise the point, which action, combined with that of the rotation, causes the point to move first to one side (the right for right-handed rotation) and then downward. This movement, by virtue of which the axis of the projectile tends to describe a cone about the tangent to the trajectory, is called "precession," and its result, in combination with the angular motion of the tangent caused by the curvature of the trajectory, is to keep the point of the projectile always on

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that side of the plane of fire towards which it was first deflected.* Thus, with right-handed rifling, the projectile during its flight always points very slightly to the right of the direction of motion and, as a result of this, the resistance of the air has a component normal to that direction which carries the projectile bodily to the right with increasing velocity.†

91. The precise determination of the drift is a matter of great difficulty, as its value is materially affected by lateral wind pressure and by the unavoidable differences between the projectiles. Experience indicates, however, that for similar projectiles it is proportional to the tangent of the final inclination of the rifling and to the square of the vertical component of the initial velocity. Thus, for United States naval guns, which use practically similar projectiles, and all but a few early designs of which have the same final twist of rifling, one turn in 25 calibers, we may write:

$$D = kV^2 \sin^2 \alpha \qquad (74)$$

in which D is the drift in feet, for the initial velocity V and the angle of departure α, and the value of k for common shell is about .0025. For A. P. shell, which are shorter than the common shell, the value of k is probably slightly greater.

It will be seen, therefore, that the drift of high-power guns at moderate ranges is very small and only becomes important in long-range firing, that of the 12-inch gun (V = 2800 f. s.), for example, being only about 3 yards at 3000 yards range, while it is about 110 yards at 12,000 yards range.

92. **The Theory of Sights.**—Guns are usually, and naval guns always, pointed by directing what is called the "line of sight" at

*The imprints of projectiles at their points of fall upon the ground at long ranges show this to be the case.

†This applies only to direct fire; when the angle of departure exceeds about 70°, as it sometimes does in mortar fire, the drift is reversed in direction. The reason for this appears to be that at the vertex of such a trajectory the direction of the tangent changes so suddenly that the slow movement of precession is insufficient to cause the axis of the projectile to keep pace with it; the angle between tangent and axis becomes greater than 90°; the projectile moves base first; the resistance of the air acts upon the opposite side of the projectile to that upon which it acted in the ascending branch; and the lateral velocity of the projectile is speedily checked and reversed. With these very high angle trajectories the projectile always strikes butt first

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the target. Originally the upper surface of the gun itself was used as the line of sight; this was called "sighting by the line of metal," and resulted in the gun having an angular elevation corresponding to the difference in thickness of the metal at breech and at muzzle, which elevation determined the range. Later on a piece of wood or metal, called a dispart, was secured to the muzzle, so as to give a line from breech to muzzle parallel to the gun's axis. Such a line of sight had to be directed more or less above the target according to the range. Early in the last century the method of having one fixed and one movable sight, so that the line between them, which is the line of sight, can be adjusted at any desired angle with the axis of the gun, came into use. At the present time a telescope, so mounted as to be capable of being set at any desired angle with the gun's axis, is the most approved form of sight.

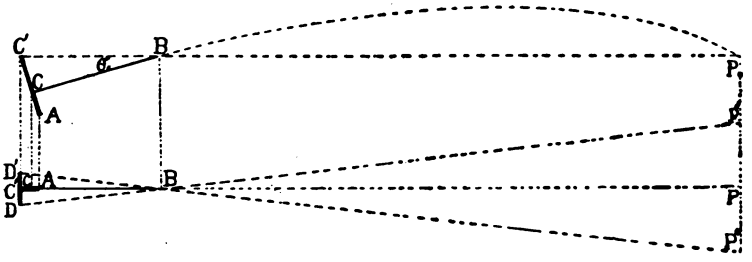

Fig. 11.

93. With ordinary sights it is customary to make the rear sight the movable one, and to graduate it with ranges (in yards or meters), with elevations (in degrees) and with the times of flight (in seconds) corresponding to the range and elevation marks. In rare instances this practice is reversed and the front sight is the movable graduated one while the rear sight is fixed. The diagram (Fig. 11) represents the usual arrangement, AC' being the movable graduated sight bar, at right angles to the axis of the gun, and B the fixed front sight. C'B is the "line of sight," being a line from the notch of the rear sight C' to the tip of the front sight B, and CB is the position of the line of sight when it is parallel to the axis of the gun, the rear sight notch having being lowered to C, usually its lowest position. The distance $CB = l$ is called the "sight radius" of the gun, and the line CB

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is sometimes called the "natural line of sight." It will be seen that when the rear sight notch is raised to C', and the line of sight C'B is directed at a target P, the axis of the gun, which is parallel to CB, is elevated the angle $C'BC = \theta$ above the target. The distance $C'C = h$ is the "sight-bar height" for the angle of elevation θ, and is evidently given by the formula

$$h = l \tan \theta \qquad (75)$$

When the heights for the range graduations of the sight bar are to be calculated, the angles of departure for the ranges are first found and then the proper values of θ to use in (75) are found by subtracting the angle of jump.

In the figure the trajectory is represented as if the axis of the gun coincided with the natural line of sight instead of being, as it is, distant from it by at least the radius of the gun's breech. No appreciable error, however, results from making this assumption in the ordinary use of guns, and it is only in proving-ground work, when firing at armor plates or other very close targets, that it is necessary to take account of the real distance which separates these two parallel lines.

94. The Sliding Leaf.—Besides the movement of the rear sight up or down to enable the gun to be pointed with the proper elevation, it is desirable to have means of moving it sidewise so that the line of sight may be adjusted at any desired angle, in the horizontal plane, with the axis of the gun, for the purpose of allowing for the drift, or other lateral deviations of the projectile, by causing the gun to point the proper amount to one side of the target at which the line of sight is directed. This is usually accomplished by forming the rear sight notch in a "sliding leaf," a piece mounted on the head of the sight bar and movable at right angles to it and to the gun's axis by suitable gearing.

In Fig. 11, D and D' represent two positions of the sliding leaf on opposite sides of its central position C'. Evidently, if the line of sight D'B is directed at a target P'', and there is no deviation of the projectile in flight, the latter will fall at P; and so a movement $C'D' = d_1$ of the sliding leaf *to the left* will cause the projectile to fall $P''P = D_1$ *to the left* of the point aimed at P''. And, similarly, moving the sliding leaf $C'D = d_2$ *to the right* will cause the projectile to fall $P'P = D_2$ *to the right* of the point aimed at P'.

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Thus to correct for a deviation of the projectile due to drift, wind, motion of gun, or any other cause, the sliding leaf is moved in the opposite direction to the deviation, and we have the general rule: *Move the sliding leaf to the side towards which you wish the projectile to go.* Also, for the relation between the motion (d) of the sliding leaf and the resulting deviation (D) of the projectile, we have, from similar triangles, $\frac{DC'}{C'B} = \frac{P'P}{PB}$, or $\frac{d}{l \sec \theta} = \frac{D}{X}$, whence

$$d = \frac{l \sec \theta}{X} D \qquad (76)$$

The error which results from putting $\sec \theta = 1$ in (76) is inappreciable for the small angles of elevation required in the ordinary use of modern guns, being only half of one per cent for $\theta = 6°$; and, as it is only a little over three per cent for $\theta = 15°$, and the value of D to be allowed for is seldom known as closely as that, it is evident that in direct fire, under all ordinary circumstances, we can use $d = \frac{l}{X} D$ in place of (76).

The graduations to which the sliding leaf is set are sometimes minutes of arc, the mark 10′ for example, being placed at a distance on each side from the central position given by $d = l \tan 10'$; and sometimes they are knots, the mark 10 knots, for example, being placed on each side at such a distance from the central position as to cause a deviation of the projectile sufficient to compensate for motion of the target across the line of fire at a speed of 10 knots, the range being some assumed average one. The best method of graduation, however, is in equal divisions, each the same fraction, preferably $\frac{1}{1000}$th, of the sight radius. With this graduation, setting the sliding leaf at the mark 10, for example, means moving it to one side the distance $d = 10 \frac{l}{1000}$ and thus compensating for a deviation of the projectile $D = 10 \frac{X}{1000}$, since, by (76), $\frac{d}{l} = \frac{D}{X}$.

95. The Permanent Angle.—It is a common practice, especially with naval guns, to correct for the drift automatically by inclining the sight bar in a plane perpendicular to the axis of the gun so that it makes with the vertical an angle i called the "permanent angle." Referring to Fig. 12, CB = l being

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the natural line of sight, CD = h is the sight bar height for the angle of elevation θ and is now given by

$$h = l \tan \theta \sec i \qquad (77)$$

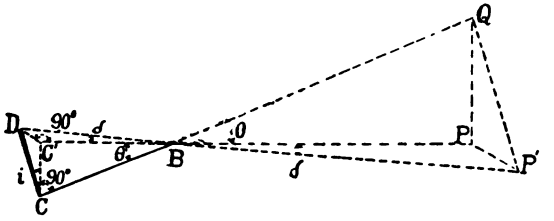

Fig. 12.

Then, if PP′ is the drift at the range X, from the similar triangles we have $\frac{DC'}{C'B} = \frac{D}{X}$; but $DC' = h \sin i = l \tan \theta \tan i$, and $C'B = l \sec \theta$. Therefore $\frac{\tan \theta \tan i}{\sec \theta} = \sin \theta \tan i = \frac{D}{X}$. Whence we have

$$\tan i = \frac{D}{X \sin \theta} \qquad (78)$$

If D were proportional to X sin θ, which it is not far from being, setting the sight bar at the angle given by (78) would exactly compensate for the drift at all ranges. Actually D increases a little more rapidly than X sin θ, and so the sight bar should be more inclined for long than for short ranges. In practice i is calculated for an assumed average fighting distance, and the bar being set at that angle the sliding leaf is either omitted or if present is only used to correct for other deviations than drift.

Evidently the bar must be inclined to the left to correct for the drift of guns with right-handed rotation, and vice versa.

96. The Telescope Sight.—The great advantage of the telescope sight is that with it the line of sight, which is its optical axis, is much more clearly defined and can be directed with much greater accuracy at any desired point than in the case of ordinary sights. The theory of the telescope sight, however, does not differ at all from that of ordinary sights. The telescope is so arranged that it can be set at any desired vertical angle with the axis of the gun, thus enabling the gun to be pointed at the proper elevation, and also that it can be moved

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in azimuth, thus giving correction for lateral deviations of the projectile. The graduations for the vertical angular motion are usually only ranges, and those for the lateral angular motion are best, as already set forth for sliding leaf graduations, such that each division corresponds to an equal increment of the tangent of the angle. Drift may be approximately corrected by inclining the axis about which the telescope turns in elevation and depression, just as it is by inclining a sight bar, and the proper angle of inclination is given by the same formula (78).

97. **The Angle of Jump.**—Jump is the increase of the angle of departure resulting from angular motion of the gun in the vertical plane caused by the shock of the discharge. This angular motion comes partly from the recoil of the gun, which moves to the rear from ½″ to 2½″, according to the caliber, before the projectile leaves the muzzle, and partly from elastic yielding of the gun carriage and platform. With modern naval guns, which recoil in the line of fire, the jump is very small, often indeterminate, and is probably almost entirely due to elastic yielding of the gun carriage. At all events in practice we assume this to be the case, since the jump determined at the proving ground is taken to be the same as the jump after installation on board ship, where the supporting platform is of an entirely different character. Another assumption which is usually made in practice is that the jump is the same at all elevations: this certainly is not true for all gun mountings, but it probably is very nearly true for modern naval mounts.

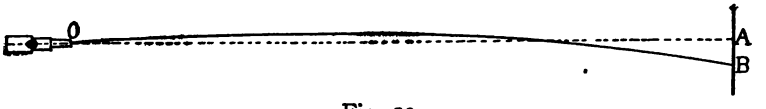

Fig. 13.

98. The angle of jump should be determined as follows: A target of planks covered with cardboard is set up at a distance of several hundred yards; the gun is laid approximately horizontally, and the intersection of its axis with the target is fixed as accurately as possible by means of bore sights. Let A (Fig. 13) be this point of intersection, and let B be the point where the projectile strikes the target when the gun is fired. The distance AB having been carefully measured for a number of shots, let its mean value be H, the distance of the target from

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the muzzle of the gun being X, and the mean muzzle velocity, which should be measured at each shot, being V. Then, taking the origin at the muzzle of the gun O, and the line of the gun's axis, OA, as the axis of X, the angle of departure is evidently the jump j, and the equation of the trajectory must be satisfied by the coordinates of B, which are $x = X$, $y = H$. Hence we have, from (24), neglecting the higher powers of kx, which are too small to be taken account of,

$$H = X \tan j - \frac{gX^2}{2V^2 \cos^2 j}(1 + \tfrac{2}{3}kX)$$

or, putting $\cos j = 1$, since j never exceeds a few minutes of arc,

$$\tan j = \frac{gX}{2V^2}(1 + \tfrac{2}{3}kX) + \frac{H}{X} \qquad (79)$$

in which H is negative when B falls below A, and the values of k are given for U. S. naval projectiles in **32**.

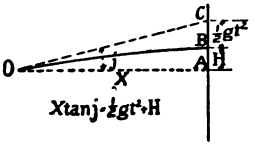

Fig. 13a.

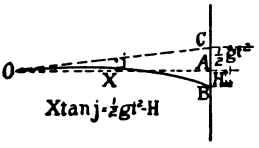

Fig. 13b.

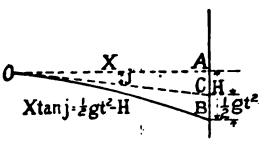

Fig. 13c.

99. If the target is set up quite close to the gun, instead of at a distance of 300 yards or more, the velocity of the projectile may be taken as constant from muzzle to target, so that the time of flight is given by $T = \frac{X}{V}$. Then, A being the intersection of the gun's axis with the target, B the point of impact of the shot, and C the intersection of the line of departure with the target, the distance $CA = X \tan j$ must be the algebraic sum of the drop of the projectile under gravity ($\frac{1}{2}gT^2$) and the measured distance of B above A (H). Whence we have

$$X \tan j = \tfrac{1}{2}gT^2 + H \qquad (80)$$

Figs. 13 (a), (b) and (c) show the three possible cases, in the first the jump and the value of H being both positive, in the second the jump being positive with H negative, and in the third both jump and H being negative.

It will be noted that (79) reduces to (80) if $\frac{2}{3}kX$ is small enough to be called zero.

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EXAMPLES IX.

(1) The 6-inch gun (V = 2900 f. s.), when fired in calm weather with angles of departure 3° 04′, 5° 03′, and 7° 02′ gave ranges 5394, 7298 and 8628 yds. and deviations, right, 22, 56 and 94 yds. Find the values of k (Eq. 74) corresponding to these results, and with its mean value calculate the drift at 2500 yds. range ($\alpha = 1° 01'$).

.00274; .00258; .00224; 6.7 ft.

(2) The mean of three rounds with the 12-inch gun, V = 2828 f. s. and $\alpha = 7° 32'$, gave 11,900 yds. range and 115 yds. drift. If the drift is proportional to $(V \sin \alpha)^2$, what is its value for V = 2800 f. s. at 2000 yds. ($\alpha = 46'$), 3000 yds. ($\alpha = 1° 12'$), and 4000 yds. ($\alpha = 1° 40'$)?

3.5; 8.6; 16.6 ft.

(3) The mean of three rounds with the 4-inch gun, V = 2000 f. s., gave 74, 127, 211 and 328 ft. drift for values of $\alpha = 5° 03'$, 7° 03′, 9° 02′ and 11° 02′. Determine the corresponding values of k (Eq. 74). .00239; .00211; .00214; .00224.

(4) Determine the sight-bar height for 2500 yds. range for the 6-inch gun of Ex. (1), there being no jump and the sight radius being 79.75″. How far and in what direction should the sliding leaf be moved to correct for the drift (6.7 ft. right) at that range? 1.415″; 0.0713″.

(5) Compute the sight-bar heights for 2000, 3000 and 4000 yds. range for the 12-inch gun of Ex. (2), sight radius = 120″ and no jump. 1.606; 2.514; 3.492 in.

(6) What should be the offsets with the sliding leaf of the 12-inch gun of Ex. (2) to correct for the drifts (3.5, 8.6 and 16.6 ft.) at the stated ranges (2000, 3000 and 4000 yds.)?

.0700; .1147; .1661 in.

(7) How much should the sight bar of Ex. (1) be inclined to correct for the drift at 2500 yds., and how much in error would the resulting correction be at the range given by $\alpha = 3° 04'$ (5394 yds. range and 22 yds. true drift)?

2° 53′; 22 ft. too small.

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(8) How much should the sight bar of Ex. (2) be inclined to correct for the drift at 4000 yds., and how much in error would the resulting correction be at the range given by $\alpha = 7°$ 32′ (11,900 yds.)? $2°\ 43'$; 122 ft. too small.

(9) The sight radius of the 6-pdr. Hotchkiss gun is 37.18″, the permanent angle of the sight bar is 2°, and the angles of elevation for 2000 and 3000 yds. range are $2°\ 31'$ and $4°\ 42'$. What are the sight-bar heights for those ranges, and how much drift is corrected at them?

1.635; 3.059 in.
9.2; 25.8 ft.

(10) A 6-pdr. gun, $V = 2100$ f. s., is fired at a target 500 yds. from the muzzle and the mean measured distance of the impact below the prolongation of the axis of the bore is 78.0″. Determine the jump (k for 6-pdr. is .00011). $+6'$.

(11) If the jump of a 4-inch gun ($V = 2100$ f. s.) is $+6'$, where will its projectile strike a screen 100 feet from the muzzle? $H = +1.656''$.

(12) An 8-inch gun, $V = 1700$ f. s., is fired at a screen 106 ft. from the muzzle and the projectile strikes in the prolongation of the axis of the bore. What was the jump? $+2'$.

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CHAPTER X.

THE EFFECT OF VARIATIONS OF THE BALLISTIC ELEMENTS UPON THE RANGE.

100. The data contained in range tables are only correct for standard conditions, and it is often necessary to determine the effect of variations of the elements upon which they are based, such as the initial velocity, the angle of departure, and the ballistic coefficient.

In deducing the relation between these variations and the resulting change of range, we assume that the effect of each cause is independent, so that the combined effect of a number of variations is merely the sum of the effects separately considered.

101. The Effect of a Variation of the Angle of Departure.— In Chapter III it has been shown that the relation between the range in air and the angle of departure for flat trajectories, is given by

$$\sin 2\alpha = \frac{gnX}{V^2} = \frac{g}{V^2} X (1 + \tfrac{2}{3}kX) \qquad (81)$$

Taking logarithmic differentials, that is differentiating and then dividing by the original equation, α and X being the only variables, we get

$$\frac{2d\alpha}{\tan 2\alpha} = \frac{1 + \frac{4}{3}kX}{1 + \frac{2}{3}kX} \cdot \frac{dX}{X} \qquad (82)$$

But $1 + \frac{2}{3}kX = n$, whence $1 + \frac{4}{3}kX = 2n - 1$, and so (82) becomes

$$\frac{2d\alpha}{\tan 2\alpha} = \frac{2n - 1}{n} \cdot \frac{dX}{X}$$

$$\frac{dX}{X} = \frac{2n}{2n - 1} \cdot \frac{d\alpha}{\tan 2\alpha} \qquad (83)$$

And (83) gives the change in range (dX) which will result from any small change in the angle of departure ($d\alpha$) in terms of X, α and n, which is the ratio of the range in vacuum to that in air. If α is less than 5°, less than one per cent of error will

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result from putting 2α for $\tan 2\alpha$, thus making the relation $\frac{dX}{X} = \frac{n}{2n-1} \cdot \frac{d\alpha}{\alpha}$, and even with α as great as 10° the error from this substitution will be but 4 per cent.

As an example, we will find the change of range in the case of an 8-inch gun, with an I. V. of 2300 f. s. and giving 2000 yds. range for $\alpha = 1° 11'$, if there is a change of 4′ in α. We first find n from $n = \frac{V^2 \sin 2\alpha}{gX} = 1.131$; then $dX = \frac{nX}{2n-1} \frac{d\alpha}{\alpha}$ $= \frac{1.131 \times 6000}{1.262} \times \frac{4}{71} = 303$ ft $= 101$ yds. So an increase, or diminution, of the angle of departure by 4′ will increase, or diminish, the range by 101 yds.*

This particular relation is much more easily obtained directly from the range table, which in the case of this 8-inch gun gives $\alpha = 1° 07'$ for X = 1900 yds. and $\alpha = 1° 15'$ for X = 2100 yds., but the algebraic expression (83) for the relation has other uses, as will be seen further on.

102. If the impact upon a vertical screen at the range X, instead of upon the horizontal plane, be considered, the change of height of impact (dy) will be related to the corresponding change of range (dX) by the equation

$$dy = dX \tan \omega \qquad (84)$$

But by (30) $\tan \omega = \frac{2n-1}{n} \tan \alpha$, so that (83) may be written

$$\frac{dX}{X} = \frac{2 \tan \alpha}{\tan \omega} \frac{d\alpha}{\tan 2\alpha}$$

whence, putting $\frac{2 \tan \alpha}{1 - \tan^2 \alpha}$ for $\tan 2\alpha$, and dy for $dX \tan \omega$, we get

$$dy = (1 - \tan^2 \alpha) X d\alpha \qquad (85)$$

Were the principle of the rigidity of the trajectory exact, we should have $dy = X d\alpha$, and so we see that the assumption of that principle amounts to neglecting $\tan^2 \alpha$ in comparison with unity.

*The change of α ($d\alpha$) must of course be expressed in circular measure (1′ = .0002909) in equation (83) and other similar formulæ, but the ratio $\frac{d\alpha}{\alpha}$ has the same value when $d\alpha$ and α are expressed in minutes of arc as when they are expressed in circular measure.

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103. The Effect of a Variation of the Initial Velocity.— Writing (81) in the form $V^2 \sin 2\alpha = gX(1 + \frac{2}{3}kX)$ and taking logarithmic differentials, V and X being now the variables, we get

$$\frac{2dV}{V} = \frac{1 + \frac{4}{3}kX}{1 + \frac{2}{3}kX}\frac{dX}{X} \tag{86}$$

which, substituting n for $1 + \frac{2}{3}kX$ and $2n - 1$ for $1 + \frac{4}{3}kX$, becomes

$$\frac{dX}{X} = \frac{2n}{2n - 1} \cdot \frac{dV}{V} \tag{87}$$

and (87) gives the change of range (dX) which will result from any small change of initial velocity (dV).

Comparing (87) with (83) it will be seen that any relative change of V produces just twice as great a change of X as would the same relative change of α.

To find the change in height of impact (dy) at range X due to the change dV, we put $\frac{\tan \alpha}{\tan \omega}$ for $\frac{n}{2n - 1}$ and dy for $\tan \omega dX$ in (87), thus getting

$$dy = 2X \tan \alpha \frac{dV}{V} \tag{88}$$

As an illustration of the use of (87) and (88), we will find the change of range and height of impact caused by a change of 50 f. s. in the muzzle velocity of a 6-inch gun which, with 2400 f. s. I. V., gives 3200 yds. range for $\alpha = 2° 06'$. Here we have $n = \frac{2400^2 \sin 4° 12'}{32.2 \times 9600} = 1.365$: then $\frac{dX}{X} = \frac{2.730}{1.730} \cdot \frac{50}{2400}$ $= \frac{1.578}{48}$: whence $dX = 9600 \times \frac{1.578}{48} = 315.6$ feet. Also we have $dy = 19200 \times \frac{50}{2400} \tan 2° 06' = 400 \times .03667 = 14.67$ feet. Thus a change of 50 f. s. in the muzzle velocity will, in this case, change the range 105 yards and change the height of impact at 3200 yards 14.67 feet.

A comparison of results given by (87) with the differences of ranges calculated for different values of V by the use of the ballistic tables will show a close agreement.*

*In Nicol's valuable and interesting "Traité d'Artillerie a l'usage des Officiers de Marine" the relation $\frac{dX}{X} = \frac{2}{2n - 1} \cdot \frac{dV}{V}$ is deduced, the omission of the factor n being due to the erroneous assumption that n varies with V independently of variations of X.

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104. The Effect of a Variation of the Ballistic Coefficient.— In this case k, which depends upon the ballistic coefficient, and X are the only variables. Therefore, differentiating $X(1+\frac{2}{3}kX) = \frac{V^2 \sin 2\alpha}{g}$, we have

$$dX + \tfrac{4}{3}kX\,dX + \tfrac{2}{3}X^2 dk = 0$$

$$\frac{dX}{X} = -\frac{\frac{2}{3}X}{1+\frac{4}{3}kX}\,dk \qquad (89)$$

But $\frac{2}{3}X = \frac{n-1}{k}$ and $1+\frac{4}{3}kX = 2n-1$, hence (89) becomes

$$\frac{dX}{X} = -\frac{n-1}{2n-1}\,\frac{dk}{k} \qquad (90)$$

or, since $k = \frac{A}{C'}$ (see 32), $\frac{dk}{k} = -\frac{dC'}{C'}$,

$$\frac{dX}{X} = \frac{n-1}{2n-1}\,\frac{dC'}{C'} \qquad (91)$$

And (91) gives the change in the range (dX) which results from any small change in the ballistic coefficient (dC').

For example, we will determine the effect upon the range in the case of a 6-inch gun for which V = 2900 f. s., $\alpha = 1° 17'$, and X = 3000 yards, if the radius of the ogival head of its projectile is changed from 2 calibers (i = 1) to 2½ calibers (i = 0.91).* Here $n = \frac{V^2 \sin 2\alpha}{gX} = 1.300$, and since $C' = \frac{w}{i\delta d^2}$, $\frac{dC'}{C'} = -\frac{di}{i} = +\frac{.09}{1.00} = .09$. Therefore we have $\frac{dX}{X} = \frac{.3}{1.6} \times .09 = .0169$; dX = .0169X = 152 feet; and the range will be increased 152 feet by the change of shape of the projectile.

By substituting $\frac{\tan\omega}{\tan\alpha}$ for $\frac{2n-1}{n}$ in (91), and then putting dy in place of tan ω dX, we find for the change of height of impact at range X caused by the change dC',

$$dy = \frac{n-1}{n} X \tan\alpha \frac{dC'}{C'} \qquad (92)$$

105. The Effect of a Variation of the Projectile's Weight.— Here we have to take account of a double effect, since both the ballistic coefficient and the initial velocity are changed when the

* This value of i for the more pointed head is not known to be correct, but is assumed as approximately so.

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weight of the projectile is changed, the weight of powder charge being supposed to remain unaltered.

Since $C' = \frac{w}{\delta i d^2}$, $\frac{dC'}{C'} = \frac{dw}{w}$, and so the change in X caused by changing w, provided V remained unchanged, would be given by $\frac{dX}{X} = \frac{n-1}{2n-1}\frac{dw}{w}$. But the change in w causes a change in V given approximately by $\frac{dV}{V} = -\frac{4}{10}\frac{dw}{w}$, and this, by (87), would of itself produce a change of range given by $\frac{dX}{X} = \frac{2n}{2n-1}\frac{dV}{V} = -\frac{.8n}{2n-1}\frac{dw}{w}$. Hence we have for the total change of range due to the change dw,

$$\frac{dX}{X} = \frac{n-1}{2n-1}\cdot\frac{dw}{w} - \frac{.8n}{2n-1}\frac{dw}{w}$$

$$\frac{dX}{X} = \frac{.2n-1}{2n-1}\frac{dw}{w} \qquad (93)$$

Since the value of n rarely exceeds 3, and probably never reaches 5, (93) seems to show that increasing a projectile's weight always diminishes its range, for the same powder charge and angle of departure.

As an example we will determine the effect upon the range of a variation of ± 5 per cent in the weight of the projectile of a 6-inch gun of 2400 f. s. I. V. laid for a range of 3000 yards ($\alpha = 1^\circ\ 55'$). We first compute the value of $n = \frac{V^2 \sin 2\alpha}{gX} = 1.329$; then, from $\frac{dw}{w} = +.05$, $\frac{dX}{X} = -\frac{.734}{1.658} \times .05 = -.02214$. and so $dX = -9000 \times .02214 = -199$ feet. Therefore an increase of weight of 5 per cent will diminish the range 199 feet, while a decrease of weight of 5 per cent will increase the range by the same amount.

Putting $\frac{n \tan \omega}{\tan \alpha}$ in place of 2n — 1, and dy in place of $\tan \omega dX$, in (93), we obtain the following expression for the change of height of impact (dy) at range X caused by the change of weight of projectile (dw),

$$dy = \frac{.2n-1}{n} X \tan \alpha \frac{dw}{w} \qquad (94)$$

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106. The Effect of a Variation of the Sight-bar Height.—Since by (75) $h = l \tan \theta$, we have by differentiation $dh = l \sec^2 \theta d\theta$, or $d\theta = \frac{dh}{l} \cos^2 \theta$. But α only differs from θ by the constant angle of jump, so that $d\alpha = d\theta$; hence, by substituting $\frac{dh}{l} \cos^2 \theta$ for $d\alpha$ in (85), we obtain for the change of height of impact (dy) at range X due to a variation dh in the height of the sight bar,

$$dy = X(1 - \tan^2 \alpha) \cos^2 \theta \frac{dh}{l}$$

in which, since θ differs very little from α, we can put $\cos^2 \alpha$ for $\cos^2 \theta$, thus obtaining

$$dy = X(\cos^2 \alpha - \sin^2 \alpha) \frac{dh}{l}$$

$$dy = X \cos 2\alpha \frac{dh}{l} \qquad (95)$$

If the trajectory is taken to be rigid, we have $dy = X d\alpha = X \cos^2 \theta \frac{dh}{l}$, or, when θ is small,

$$dy = X \frac{dh}{l} \qquad (96)$$

To find the change of range caused by a variation of the sight-bar height (dh), we have only to multiply the value of dy given by either (95) or (96) by $\cot \omega = \frac{n \cot \alpha}{2n - 1}$.

EXAMPLES X.

(1) A Krupp 28 cm. (11.024″) gun, V = 1733 f. s., w = 760.4 lbs., gave a range of 6780 ft. for $\alpha = 2° 23'$. What would the range have been for $\alpha = 2° 11'$? 6274 ft.

(2) The gun of Ex. (1) gave a range of 20,364 ft. for $\alpha = 8° 30'$. What would the range have been for $\alpha = 8°$? 19,436 ft.

(3) The angles of departure for the 12-inch gun, V = 2250 f. s., w = 850 lbs., for 2000, 3000 and 4000 yds. range are respectively $1° 11'$, $1° 52'$ and $2° 36'$. Determine the change of range which an error of $1'$ in laying the gun for each of those ranges will cause. 78.6; 71.7; 66.5 ft.

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(4) What will the change of height of impact be in each of the three cases of Ex. (3)? 1.75; 2.62; 3.48 ft.

(5) The 6-inch gun of 30 calibers length (V = 2000 f. s.) gives 2500 yds. range for $a = 2° 13' 24''$. What will be the range of the 35 caliber gun, for which V = 2075 f. s., at the same angle of departure? How much higher would the impact on a vertical screen at 2500 yds. range be in the second than in the first case? 2654 yds.; 21.84 ft.

(6) The 12-inch gun of 2800 f. s. I. V. being laid for 3000 yds. range ($a = 1° 12'$), what will be the decrease of range if the I. V. is only 2750 f. s.? How much lower will the impact be on a vertical screen at 3000 yds.? 96 yds.; 6.73 ft.

(7) The Krupp 24 cm. (9.45") gun gave a range of 19,567 ft. for V = 1736 f. s. and $a = 8° 35'$. How much less would the range have been with V = 1700 f. s.? 628 ft.

(8) The Krupp 40 cm. (15.75") gun gave a range of 39,808 ft. for V = 1805 f. s. and $a = 18°$. How much greater would the range have been for V = 1850 f. s.? 1492 ft.

(9) The 10-inch gun of 30 calibers length (V = 2000 f. s.) gives 3000 yds. range for $a = 2° 28'$. What will be the range of the 34 caliber gun, for which V = 2060 f. s., at the same elevation? 3156 yds.

(10) The 6 pds. range table (for V = 1870 f. s.) gives $a = 2° 31'$ for 2000 yds. range. If, instead of being of standard density, the air is at 32° F. with barometer at 30.00", what will the range be for $a = 2° 31'$? How much will the change of density of the air lower the point of impact at 2000 yds. range? 1960 yds.; 7.23 ft.

(11) The 6-inch gun of 2900 f. s. I. V. is laid for a range of 2000 yds. at the range table angle of elevation ($a = 0° 47'$). The thermometer being at 84° and the barometer 28.75", how far over will the gun carry? What will be the increase of height of impact at 2000 yds. range? 62 ft.; 0.98 ft.

(12) The 3-inch gun with w = 13 lbs., and V = 2800 f. s. gives a range of 2000 yds. with $a = 1° 00' 43''$. If a 15 lb. projectile is substituted for the 13 lb. one, the powder charge and

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elevation remaining unchanged, what will the range be? (Compare result with Examples VII (19) and (20).)

1882 yds.

(13) A 5-inch gun with $w = 60$ lbs. and $V = 2900$ f. s. gives a range of 3000 yds for $\alpha = 1^\circ\ 21'\ 16''$. What will the range be with a 55 lb. projectile, the powder charge and elevation remaining unchanged? (Compare result with Examples VII (21) and (22).) 3095 yds.

(14) The Krupp 28 cm. gun with $w = 760.4$ lbs. and $V = 1733$ f. s. gave a range of 6274 ft. for $\alpha = 2^\circ\ 11'$. What is the range when a projectile weighing 652.4 lbs. is fired with the same charge and elevation? 6819 ft.

(15) The Krupp 28 cm. gun with $w = 760.4$ lbs. and $V = 1733$ f. s. gave a range of 19,436 ft. for $\alpha = 8^\circ$. What is the range when the 652.4 lb. projectile is used with the same charge and elevation? 20,669 ft.

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CHAPTER XI.

THE EFFECT OF WIND, OF MOTION OF GUN, AND OF MOTION OF TARGET.

107. In considering the effect of wind upon the flight of a projectile, we are obliged, for want of better knowledge, to assume that the air moves only horizontally and that its velocity and direction are the same throughout the trajectory as we observe them to be at the gun. Actually the wind is never steady, either in force or direction; its velocity usually increases with the height above the ground, and its motion is not always confined to a horizontal plane. Moreover, lateral wind pressure alters the drift due to rotation, a wind from the left causing the axis of the projectile to hug the trajectory more closely and thus reducing the drift (when the rotation is right-handed), and a wind from the right tending to increase the drift (with right-handed rotation).

It is for these reasons that the deviations caused by the wind can only be roughly approximated to, and, consequently, that experiments for determining any of the ballistic constants, to be of value, must be made when it is calm or very nearly so.

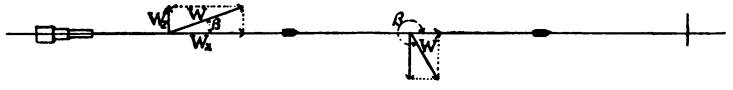

Fig. 14.

108. We shall denote by W the velocity of the wind in feet per second, and by W_x and W_z, respectively, the components of that velocity in and at right angles to the plane of fire; we shall call W_x positive when it is with, and negative when it is against the flight of the projectile; and we shall call W_z positive when it tends to carry the projectile from right to left of an observer looking from gun to target and negative in the opposite case. Then, referring to Fig. 14, if we denote by β the angle between the direction from gun to target and the direction towards which the wind is blowing, measuring that angle to the left from the

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first direction round to the second, we shall always have the wind components given with their proper signs by the formulæ

$$\left.\begin{aligned} W_x &= W\cos\beta \\ W_z &= W\sin\beta \end{aligned}\right\} \qquad (97)$$

The respective values of the wind components being thus found, they are treated as independent of one another, each producing its own effect.

109. The Effect of Wind upon the Range.—Let W_x be the wind component in the plane of fire in f. s., plus when with the flight of the projectile, and minus when against it; let V and α be the initial velocity and angle of departure relative to the stationary ground; and let X be the range if there were no wind.

If now we combine W_x reversed in direction with V, the resultant will be the initial motion of the projectile *relative to the air,* and the range corresponding to this resultant (which we will call V′ and its angle with the horizontal α') will be the range *relative to the air* (which we will call X′). Then, if T be the time of flight, the range over the ground will evidently be $X' + W_x T$, since, while the projectile moves X′ through the air, the air itself moves $W_x T$ over the ground. Consequently the increase of range due to the wind W_x is given by

$$\Delta X = X' + W_x T - X \qquad (98)$$

in which X and X′ are the ranges *in still air* corresponding respectively to V, α and V′ α', and T is the time of flight of the latter trajectory.

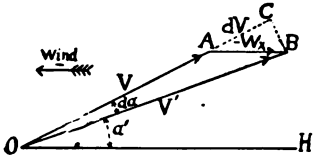

Fig. 15a.

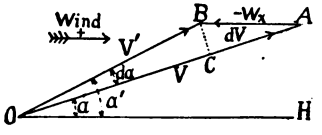

Fig. 15b.

In Fig. 15 (a) is represented the case where W_x is negative, and it will be seen that $OA = V$ combined with $AB = -W_x$ gives $OB = V'$, which is *greater* than V by $dV = AC = W_x \cos\alpha$, while the angle $BOH = \alpha'$ is *less* than $AOH = \alpha$ by $d\alpha = \frac{BC}{OB} = \frac{W_x \sin\alpha}{V'}$. Similarly, in Fig. 15 (b) is represented the case

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where W_x is positive, and here V′ is *less* than V by $dV = AC = W_x \cos \alpha$ while α' is *greater* than α by $d\alpha = \frac{BC}{OB} = \frac{W_x \sin \alpha}{V'}$. Thus in both cases, if W_x is given its proper sign, we obtain the values of the changes in V and α with their proper signs by the formulæ

$$\left.\begin{aligned} dV &= -W_x \cos \alpha \\ d\alpha &= \frac{W_x \sin \alpha}{V'} \end{aligned}\right\} \quad (99)$$

To determine the effect of a wind W_x, therefore, we calculate dV and $d\alpha$ by (99); calculate the range X given by V and α, the range X′ given by $V' = V + dV$ and $\alpha' = \alpha + d\alpha$; and the time of flight for the latter range; and then by (98) we obtain the change of range due to the wind.

110. An examination of Fig. 16 may perhaps help to an understanding of the foregoing method of determining the effect of wind upon the range. The case represented is one where the wind is with the flight, or W_x is plus; O represents the stationary gun and M the stationary target, the distance $OM = X$ being the range which would be given if there were no wind, the initial velocity being V and the angle of departure α. Then OSH represents the trajectory *relative to the air,* that is the

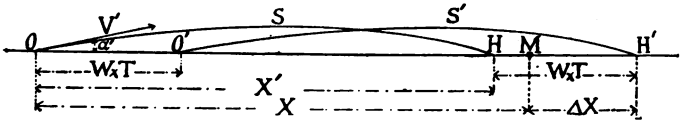

Fig. 16.

trajectory which would be given in still air if the initial velocity were V′ and the angle of departure α'. But since the air itself is moving with the velocity W_x, when the projectile reaches the ground the trajectory OSH has moved to the position O′S′H′, and the actual range over the ground is $OH' = X' + W_xT$. The projectile, therefore, instead of striking at the target M, strikes $\Delta X = X' + W_xT - X$ beyond it.

111. A formula for ready use, and one as likely to give correct results as the foregoing method, provided α be not too large, is obtained by substituting in (98) the value of $X' - X$ as given

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by the formulæ deduced in the last chapter. The difference $X' - X$, as already explained, is nothing but the change of range which would result from increasing V by dV and α by $d\alpha$, dV and $d\alpha$ being determined by (99), but by (87) $\frac{dX}{X} = \frac{2n}{2n-1} \cdot \frac{dV}{V}$, and by (83) $\frac{dX}{X} = \frac{2n}{2n-1} \frac{d\alpha}{\tan 2\alpha}$, and so the change of X due to the simultaneous changes $dV = -W_x \cos \alpha$ and $d\alpha = \frac{W_x \sin \alpha}{V'}$ is given by

$$\frac{X' - X}{X} = \frac{2n}{2n-1}\left(\frac{W_x \sin \alpha}{V' \tan 2\alpha} - \frac{W_x \cos \alpha}{V}\right)$$

and putting V for V' in this, which we can do without material error because dV is always very small compared with V, this reduces to

$$\frac{X' - X}{X} = \frac{2n}{2n-1} \cdot \frac{W_x}{V}\left(\frac{\sin \alpha}{\tan 2\alpha} - \cos \alpha\right)$$

$$\frac{X' - X}{X} = \frac{n}{2n-1} \frac{W_x \cos \alpha}{V}\left(\frac{2 \tan \alpha}{\tan 2\alpha} - 2\right)$$

$$\frac{X' - X}{X} = \frac{n}{2n-1} \cdot \frac{W_x \cos \alpha}{V}(-\tan^2 \alpha - 1)$$

whence, neglecting $\tan^2 \alpha$ in comparison with unity,

$$X' - X = -\frac{n}{2n-1} \cdot \frac{X \cos \alpha}{V} W_x$$

and, substituting this in (98), we finally get for the change of range due to the wind W_x

$$\Delta X = W_x\left(T - \frac{n}{2n-1} \cdot \frac{X \cos \alpha}{V}\right) \qquad (100)$$

in which cos α may be taken as unity, and T, though really the time of flight for V' and α', may be taken as the time of flight for the actual firing data V and α, without introducing any material errors.

112. As an example of the use of (100) we will determine the effect of a 10-mile breeze in the line of fire upon the 12-inch army projectile, w = 1000 lbs., V = 2250 f. s., when fired with elevation for 8000 yards range. The range table gives $\alpha = 6° 0.5'$ and T = 13.44 seconds for this range.

Here we have $n = \frac{V^2 \sin 2\alpha}{gX} = \frac{2250^2 \sin 12° 01'}{32.2 \times 24000} = 1.364$;

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whence we find $\frac{n}{2n-1}\frac{X\cos\alpha}{V} = 8.37$; and so $\Delta X = W_x$ $(13.44 - 8.37) = 5.07\ W_x$; or, since $W_x = \frac{44 \times 10}{30} = 14.67$ f. s., $\Delta X = 74.4$ feet; and such a wind would increase or diminish the range by 25 yards according as it was with or against the flight.

113. The Lateral Deviation Caused by Wind.—Let W_z be the lateral wind component in f. s., plus when it blows from right to left across the line of fire and minus when it blows from left to right; let V and α be the initial velocity and angle of departure relative to the ground; and let X be the range if there were no wind. Then, proceeding as before, and combining W_z reversed in direction with V, we obtain the initial velocity and direction *relative to the air.* Thus, referring to Fig. 17 (a), which represents the case of a negative wind, the resultant of $OA = V$ with $OC = -W_z$ is $OB = V'$, which is very slightly greater than V; the angle $BOD = \alpha'$, which V′ makes with the horizontal, is very slightly less than $AOE = \alpha$; and V′ is inclined to the left so as to make with V the small angle BOA.

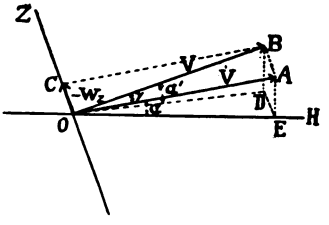

Fig. 17a.

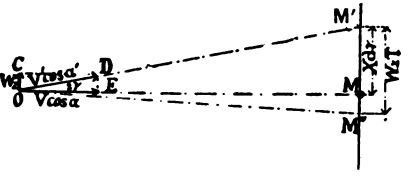

Fig. 17b.

Now, since V′ and α' differ so little from V and α, and since the effect of the increase of V is offset by that of the decrease of α, we may take the range X′ corresponding to V′, α' to be practically the same as the range X corresponding to V, α. Therefore the only essential difference between the trajectory *relative to the air* and the trajectory *relative to the ground* is that the plane of the former makes the angle $DOE = \gamma$ with the plane of the latter. Referring now to Fig. 17 (b), in which O represents the gun and M_0 the target at the range $OM_0 = X$, we see that $\tan\gamma = \frac{W_z}{V\cos\alpha}$; and OM′ is the

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horizontal trace of the trajectory *relative to the air.* But while the projectile moves through the air from O to M′, the air itself has moved $W_x T$ to the right, and so the projectile really strikes the ground at M″, to the right of the target by the distance $M_0M'' = W_x T - X \tan \gamma = W_x\left(T - \frac{X}{V \cos \alpha}\right)$. Thus the lateral deviation caused by the wind component W_x is given by the formula

$$D = W_x\left(T - \frac{X}{V \cos \alpha}\right) \qquad (101)$$

in which T, though really the time of flight corresponding to V′, α', may without appreciable error be taken as the time of flight for the actual firing data, V, α; and cos α may be taken as unity for flat trajectories.

114. As an illustration, we will determine the lateral deviation caused by the wind component $W_x = 14.7$ f. s. (10 m. h.) in the case of the U. S. rifle, caliber 0.30, at a range of 1000 yds., for which the range table gives $\alpha = 1° 36'$ and $T = 2.59$ seconds, V being 2000 f. s. Here we have $\frac{X}{V \cos \alpha} = \frac{X}{V} = \frac{3000}{2000} = 1.50$; $T - \frac{X}{V \cos \alpha} = 2.59 - 1.50 = 1.09$; and so $D = 14.7 \times 1.09 =$ 16 feet.

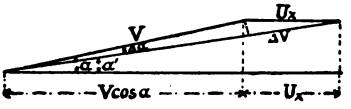

Fig. 18.

115. The Effect of Horizontal Motion of the Gun.

As in the case of wind, we resolve the horizontal velocity of the gun, due to the ship's motion, into two components, U_x in the plane of fire, and U_z at the right angles to that plane, and determine their effects separately, the first affecting the range only and the second the lateral deviation only.

116. Motion in the Plane of Fire.—Let U_x be the resolved part of the speed in the line of fire, in f. s., plus when with and minus when contrary to the flight of the projectile. Then evidently the true initial velocity of the projectile is the resultant of U_x and V, and, as shown by Fig. 18, V receives the increment $\Delta V = U_x \cos \alpha$ while α is diminished by $\Delta \alpha = \frac{U_x \sin \alpha}{V}$.

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But, by equations (87) and (83), these two changes, of V and α respectively, will cause a change of range given by

$$\frac{\Delta X}{X} = \frac{2n}{2n-1}\left(\frac{U_x \cos\alpha}{V} - \frac{U_x \sin\alpha}{V \tan 2\alpha}\right)$$

$$\frac{\Delta X}{X} = \frac{n}{2n-1} \cdot \frac{U_x \cos\alpha}{V}\left(2 - \frac{2\tan\alpha}{\tan 2\alpha}\right)$$

which, when α is small enough to make $\tan^2\alpha$ negligible in comparison with unity, reduces to

$$\frac{\Delta X}{X} = \frac{n}{2n-1} \cdot \frac{U_x \cos\alpha}{V}$$

$$\Delta X = \frac{n}{2n-1} \cdot \frac{X \cos\alpha}{V} U_x \qquad (102)$$

and cos α, of course, may be taken as unity in the case of a flat trajectory.

As an illustration we will find the effect upon the range of 15 knots speed in the case of a 12-inch gun firing directly ahead, given V = 2250 f. s., X = 4000 yards, and $\alpha = 2°\ 32.5'$. Here we have $n = \frac{2250^2 \sin 5°\ 05'}{32.2 \times 12000} = 1.161$; $U_x = 25$ f. s.; and so, taking cos $\alpha = 1$, $\Delta X = \frac{1.161}{1.322} \cdot \frac{12000}{2250} \times 25 = 117$ feet, and, of course, in the direction of the motion U_x.

117. **Motion Across the Line of Fire.**—Let U_z be the resolved part of the speed at right angles to the line of fire in f. s. Then, in addition to its initial velocity V in the line of the gun's axis, the projectile in leaving the gun has a lateral velocity U_z, and so, as may be seen from Fig. 17 (b), the real plane of departure makes an angle $\left(\gamma = \tan^{-1}\frac{U_z}{V \cos\alpha}\right)$ with the vertical plane of the gun's axis, and the resulting deviation at range X is given by $D = X \tan\gamma$, or

$$D = \frac{X}{V \cos\alpha} U_z \qquad (103)$$

Thus, for example, if the 12-inch gun of 2250 f. s. I. V. be fired abeam, the range being 4000 yds., for which $\alpha = 2°\ 32.5'$, and the ship steaming 15 kts., the deviation resulting from the motion will be (taking cos $\alpha = 1$) $D = \frac{12000 \times 25}{2250} = 133$ ft., and of course, in the direction of the motion U_z.

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118. The formulæ just deduced, (102) and (103), give the total effect upon the range and deviation of the motions U_x and U_z respectively, but it is just as correct and much simpler to proceed as follows: The horizontal velocity of the ship, which is added to the projectile's proper velocity, would add to the range the distance U_xT and to the deviation the distance U_zT, T being the time of flight, if it were not for the retardation caused by the resistance of the air. But the reduction of these added distances by air resistance is exactly equal to the change of range and to the lateral deviation which would be caused by wind components respectively equal to U_z and U_x.* Consequently, if, in determining the wind effect, we take account of the force and direction of the wind relative to the ship, that is of the *apparent* and not the real wind, in so doing we are including part of the effect of the ship's motion, and the remaining effect of that motion is to be found, not by (102) and (103), but by

$$\left.\begin{aligned}\Delta X &= U_xT\\ D &= U_zT\end{aligned}\right\} \qquad (104)$$

119. **The Effect of Motion of the Target.**—Evidently, if the target is moving in the line of fire with the velocity u_x, in order to hit it the sight bar should be set for a range greater or less than the true range at the instant of firing by the distance which the target will traverse in the time of flight, or u_xT. So, also, if the speed of the target at right angles to the plane of fire is u_z, the shot will fall u_zT to one side of the target unless that much deviation is allowed for in the pointing. Thus for the effect of motion of the target we must correct the range and deviation by the quantities given by

$$\left.\begin{aligned}\Delta X &= u_xT\\ D &= u_zT\end{aligned}\right\} \qquad (105)$$

120. As indicated by equations (104) and (105), provided the

*Observe that the change of range due to the apparent wind which results from the ship's motion U_x is by (100) $- U_x\left(T - \frac{n}{2n-1}\frac{X\cos\alpha}{V}\right)$ so that, if we take account of this apparent wind, we must use $+ U_xT$ to correct the range for the motion U_x in order that the sum of the two corrections may be the true effect of the motion as given by (102), and similarly with lateral motion.

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apparent wind, as is most convenient and usual, is used for determining the wind correction, the corrections for motion of gun and of target may be found as a single correction by merely multiplying the speed of the target *relative to the gun* by the time of flight, and this is the best and simplest way.

The speed at which the target is crossing the line of fire, itself regarded as stationary, multiplied by the time of flight gives the distance which the gun should be pointed to that side of the target towards which it appears to be moving. And, similarly, the speed with which the target is approaching, or receding, in the plane of fire multiplied by the time of flight gives the distance to be subtracted from, or added to, the true range to give the proper sight-bar setting.*

EXAMPLES XI.

(1) The .30 caliber small arm, V = 2000 f. s., is fired with an angle of departure (1° 36′) which in still air would give 1000 yds. range and T = 2.59 sec. How much short will the bullet strike if a head wind of 30 m. h. is blowing? 72 ft.

(2) The 3-inch projectile, V = 2800 f. s., ranges 2000 yds. in still air for $\alpha = 1° 1'$ with a time of flight of 2.79 sec. How much will a 30 m. h. breeze directly with the flight increase the range? 51 ft.

*The method herein adopted for the treatment of the problem of wind effect was first set forth, so far as I am aware, in General Didion's Traité de Ballistique, though it has been generally accepted since. It is mathematically correct only for spherical projectiles, to the motion of which the air offers a resistance which is independent of the direction of the motion. With elongated projectiles it will be seen that the initial motion *relative to the air* is not exactly in the line of the projectile's axis, so that we have no right to assume, as we do, that the flight *relative to the air* is the same when the air is moving as when it is still. It has been supposed by some writers that the lateral wind component produces the same pressure on the side of the moving projectile as it would if the projectile were stationary, and that the deviation can be computed upon that basis. If this were true the deviation would be proportional to the square of the lateral wind component, whereas it is really very much more nearly proportional to its first power. Actually the pressure is much greater when the projectile is moving at right angles to the wind current than when it is stationary, on account of the increased number of air particles which strike it.

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(3) The 12-inch projectile, V = 2250 f. s., ranges 2000 yds. in still air for $\alpha = 1° 10.6'$, and T = 2.82 sec. How much will a 30 m. h. breeze directly with the flight increase the range?

14.5 ft.

(4) The Krupp 28 cm. gun, w = 760.4, V = 1733 f. s., gave a range of 33,622 ft. for $\alpha = 17° 15'$ with a time of flight of 28 sec. There was a wind of 16.4 f. s. velocity with $\beta = 135°$. What would the range have been in still air? 33,789 ft.

(5) The gun of Ex. (4), firing with $\alpha = 8° 30'$, gave a range of 20,364 ft. with a time of flight of 14.8 seconds, there being a wind component of 13.8 f. s. with the flight. What would the range have been in still air? 20,288 ft.

(6) The 3-inch gun, V = 2800 f. s., gives 3000 yds. range for $\alpha = 1° 53'$ and T = 4.85 sec. What deviation will be caused at that range by a 15-knot breeze directly across the plane of fire?

41 ft.

(7) The 6-inch gun, V = 2900 f. s., gives 3000 yds. range for $\alpha = 1° 17'$ and T = 3.78 sec. What is the deviation at that range due to a 15-knot breeze directly across the plane of fire?

17 ft.

(8) The 12-inch gun, V = 2800 f. s., gives 3000 yds. range for $\alpha = 1° 12'$ and T = 3.52 sec. What is the deviation at that range due to a 15-knot breeze directly across the plane of fire?

8 ft.

(9) What lateral deviation did the wind cause in the case of Ex. (4)? 89 ft.

(10) Steaming 18 knots directly towards a stationary target at 3000 yds. distance, for what range should the bow 8-inch guns be elevated, given V = 2300 f. s. and the range-table values of α and T for 3000 yds. range being $1° 55'$ and 4.54 seconds, supposing that it is actually calm? If the apparent wind is to be allowed for, what range correction should be made for the speed?

2967 yds.; — 45 yds.

(11) Steaming 18 knots with a stationary target 3000 yds. distant on the port beam, what deviation would result from the

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speed in the case of the 8-inch gun of Ex. (10), supposing that it is actually calm? If the deviation caused by the apparent wind is allowed for, how much deviation due to the speed should be corrected? 117 ft. right; 136 ft. right.

(12) A torpedo-boat 2000 yds. from a stationary gun is making 24 knots speed on a course which makes 30° with the line from her to the gun. If the time of flight of a 6-pdr. shell for 2000 yds. range is 4 seconds, what allowance should be made in pointing the 6 pdr. at the boat? $D = 80$ ft.; $\Delta X = 139$ ft.

(13) The course and speed of an opposing ship relative to your own are such that the range is diminishing at the rate of 300 yds. a minute while she draws ahead of you 3° in the same time. When the range is 3000 yds. how should a starboard battery 6-inch gun, whose time of flight for that range is 3.9 sec., be pointed so as to allow for the relative motion?

31 ft. left; 59 ft. short.

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CHAPTER XII.

RANGE TABLES—THEIR PREPARATION AND USE.

121. The Constants which Form the Basis of a Range Table.—Complete and accurate knowledge of the elements of the trajectories of a gun and of the effects of their variations is essential to its efficient use, and the range table should furnish this necessary information.

The constants upon which a range table is based are the initial velocity (V), the ballistic coefficient ($C = \frac{w}{id^2}$), and the jump (j). The initial velocity, as well as the weight of the projectile, form features of the original design of each mark of gun, and, though they may be modified as a result of the preliminary firings, they are fixed quantities when the question of the sight graduations and the range-table data comes up. Of course they both vary from their assigned standard values, the amount of the variation from round to round depending upon the regularity of the powder and the care taken in the manufacture of the projectiles, in their inspection, and in the putting up of the charges, but the sights are marked and the range tables computed for the mean velocity and the mean projectile weight, and these should be identical with their fixed standard values. The coefficient of form is usually assumed to be unity, unless the shape of the projectile differs considerably from the standard; it would be well to determine it experimentally for each new projectile, but such determinations are worse than valueless unless based upon numerous firings made with the greatest care and under the most favorable circumstances. The jump should be determined by experiment for each new design of gun and gun mount, but here again only the result of numerous, carefully made tests is worthy of acceptance. The methods of measuring the coefficient of form and the jump have already been given.

122. Experimental Data Useless unless Standardized.—The values of the constants V, C and j having been decided upon, the

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computations of the range tables and the sight graduations is a simple matter, and the methods of making these computations have already been fully explained. When practicable, however, and especially if the value of the coefficient of form, or that of the jump, is at all doubtful, it is well to measure the ranges for a series of angles of elevation and to use the results as a basis for graduating the sights and as a check upon the other range table data. It should be clearly understood, in this connection, that a range table based entirely upon the ballistic tables, is far more likely to be correct—that is, to agree with the average results of firing—than is one based upon experiment, unless the experiments are made with great care, under favorable circumstances, and with all abnormal conditions either eliminated or properly allowed for.

123. The Data a Range Table should Contain.—The heading of the range table should give the caliber and "mark," or other title of the gun, its standard initial velocity and weight of projectile, the angle of jump, the sight radius, the permanent angle, and the value of the graduations of the sliding leaf. If the sight is a telescope, its permanent angle and the value of its azimuth graduations should be given.

The argument of the table should be the range, in yards, the stated ranges being every hundred yards from zero to the greatest range allowed by the gun mount.

Succeeding columns should give, for each stated range, the angle of departure, the time of flight, the angle of fall, the striking velocity, and the drift. It is also desirable to add columns giving the coordinates of the vertex, the danger space for a stated freeboard, and the thickness of armor which the A. P. projectile will perforate at the point of fall.

The same, or preferably a second, table, with the range as argument, should give, for each stated range, the change of range which would result from (1) a given variation of the initial velocity, say $\pm$ 10 f. s.; (2) a given variation of the density of the air, say $\pm$ 1 per cent; (3) a given wind component in the plane of fire, say $\pm$ 10 f. s. (6 knots); (4) motion of gun and target in the plane of fire during the time of flight at a given speed, say 6 knots. This table should also give, for each stated

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range, the drift remaining uncorrected by the permanent angle, the deviation caused by a given lateral wind component, say ± 10 f. s. (6 knots), and the deviation which would result from motion of gun and target perpendicular to the plane of fire during the time of flight at a given speed, say 6 knots. Instead of the lateral deviations themselves, the number of divisions of the sliding leaf which will correct for them may be given.

It will be observed that while the graduation of the sights and all the tabulated elements of the trajectory depend upon the values of the angles of departure for the stated ranges, so that the correct determination of the latter is all important, yet in the actual use of a gun the data concerning the angle of departure, the angle of fall, the time of flight and the striking velocity, which usually constitute the principal part of the range table, are of little or no direct value. The really important, and even essential, information is that which concerns the effect, upon the range and deviation, of variations of initial velocity or ballistic coefficient, of wind, and of motion of gun and target.

124. Experimental Ranging.—When the data for an experimental range table is to be obtained, it is important to eliminate, as far as possible, all uncertainties, and to this end the greatest care should be taken to see that the projectiles are brought to the exact standard weight and have rotating bands of standard diameter, that the powder charges are as nearly alike in weight and in other respects as practicable, and that the weather conditions are favorable. The less wind the better, and a strong breeze, especially when of variable strength and direction, should be regarded as prohibitive of ranging experiments.

The gun is laid as accurately as possible at successive angles of elevation, commonly at every other degree up to the maximum allowed by the carriage, and at least three, and preferably five or more, rounds are fired at each elevation. The velocity would better be measured at each round, but in lieu of that its mean value should be determined from three or more rounds fired just before and again just after the ranging.

If the firing is over land, the exact measurement of each range is, of course, a simple matter, but when, as is almost always the case, the firing is over water, so that the point of fall has to be fixed by angling on the splash, there should be a number of ob-

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servers, their positions should be selected with care, and the state of the weather as affecting the accuracy of the observations is important.

It is best to rely upon calculation to give the times of flight, as different observers will be found to record widely different times for the flight of the same shot. The lateral deviation, however, should be determined by measurement of the angular deflections by an observer stationed behind the gun, the plane of fire being a fixed one throughout the ranging so that the angles may all be taken from some definite line of bearing.

The readings of the barometer and of the wet and dry bulb thermometers, as well as the direction and velocity of the wind, should be recorded at the beginning of the ranging, and if any of these conditions change before the ranging is completed, the changes should be noted. The differences of level of the gun and the various points of fall are also necessary data, and when the firing is over water the changes of level due to the tide, if any, should be recorded.

125. The Reduction of Observed Ranges.—Having obtained the actual ranges corresponding to certain angles of elevation under certain observed conditions, the next step is to standardize these ranges, that is, to determine what they would be under standard conditions, and the order and method of procedure to do this will best be illustrated by an example.

The mean of three measured ranges given by a 4-inch gun at an elevation of 3° by quadrant was 3040 yards; the projectiles were of standard form and 33 pounds weight; the mean initial velocity given by five rounds fired just before the ranging was 1996 f. s.; the barometer stood at 30.06, the thermometer at 80°, and a wind of 30 m. h. was blowing directly against the flight during the ranging;* the height of the gun above the water was 12 feet, and the jump was $+ 5'$.

The height of the gun is first taken account of, and this is done, not by reducing the range, but by finding the angle of departure which would give a horizontal range equal to the measured range. In fig. 19, O represents the gun, at the height h above the point of fall B, and AB, which is practically equal to OB, is the measured

* Ranging a gun with such a wind blowing is very bad practice and this example is not one to follow.

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range X. Then by the principle of the rigidity of the trajectory, an angle of departure equal to the angle of projection $\theta + j + p$ would give a horizontal range X. But θ is the quadrant angle of elevation (3°); j is the jump (+ 5′); and $p = \tan^{-1}\frac{h}{X} = \tan^{-1}\frac{12}{9120} = 4'\ 33''$. Therefore the correct angle of departure for a horizontal range of 3040 yards is $3° + 5' + 4'\ 33'' = 3°\ 09'\ 33''$.

The next thing is to reduce the range for the wind, and this is done by (100). We first find the time of flight to be 6.13 seconds by the ballistic tables, having given V = 1996, X = 9120 ft.,

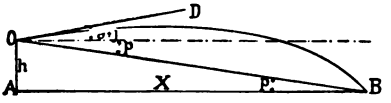

Fig. 19.

d = 4 in., w = 33 lbs., and $\delta = .975$ (δ from Table II for given barometer and temperature). We then find $n = \frac{V^2 \sin 2\alpha}{gX} = 1.493$, given V = 1996, $\alpha = 3°\ 09'\ 33''$, g = 32.2, and X = 9120. Then we find $\Delta X = W_x\left(T - \frac{n}{2n-1}\frac{X\cos\alpha}{V}\right) = 119$ ft., given T = 6.13, n = 1.493, X = 9120, $\alpha = 3°\ 09'\ 33''$, and V = 1996. And, the wind being against the flight, we add 119 to 9120 and have 9239 feet as the range in still air for the angle of departure $3°\ 09'\ 33''$.

We next reduce the range to what it would be with the standard initial velocity (2000 f. s.) by (87). With the reduced value of X (9239 ft.) we find the value of n for still air to be 1.474, and with this and dV = + 4 f. s. we find $dX = \frac{2n}{2n-1} X \frac{dV}{V} = +28$ ft. Therefore in still air and with 2000 f. s. initial velocity the range for $\alpha = 3°\ 09'\ 33''$ would be 9267 ft.

* Finally we reduce the range to what it would be in air of standard density by (91). Using the last value of n (1.474), since the last correction of the value of X would not materially

* Under ordinary circumstances it would be an unnecessary refinement to compute the exact value of δ from the reading of both wet and dry bulb thermometers. It is near enough to use the tabular value of δ.

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affect it, and having $\frac{dC'}{C'} = -\frac{d\delta}{\delta} = -\frac{.025}{.975} = -\frac{1}{39}$, we find $dX = \frac{n-1}{2n-1} X \frac{dC'}{C'} = -58$ ft. Therefore in air of standard density the range would be 9209 ft. for $\alpha = 3^\circ\ 09'\ 33''$.

Applying the same method of procedure to the other angles of elevation and measured ranges, we obtain a table of angles of departure with the horizontal ranges which would correspond to them under standard conditions.

126. From the table thus formed, the angles of departure for the stated ranges of the range table have next to be found.

A simple and rapid, though not very accurate, method of doing this is by plotting points whose abscissæ are the reduced experimental ranges and whose ordinates are the corresponding angles of departure, and drawing a fair curve through them. Then the angle of departure for any range is found by simply measuring the ordinate which corresponds to that range as an abscissa.

A much better method is by taking the equation* $\sin 2\alpha = aX + bX^2$ and determining the most probable values of a and b by least squares from the experimental values of α and X, after which the value of α for any value of X is readily calculated.

With either method it is as well only to directly determine the values of α for ranges differing by increments of 500 yards and to find the values of α for the intermediate ranges by interpolation.

127. Having thus finally arrived at the angles of departure for the stated ranges of the range table, by subtracting the angle of jump from each, we obtain the angles of elevation which will give those ranges, and from these the sight graduations are easily calculated.

It will be seen that sight graduations determined in the manner set forth are practically independent of the value which is assigned to the jump, as they are also independent of any assumption as to the value of the ballistic coefficient of the projectile, and this is the one real advantage of experimental ranging. All the other data of the range table—the time of flight, striking velocity, etc., must be calculated by means of the ballistic tables, and measured ranges only serve as a check upon values of the jump and the

*This form is suggested by the expression for the range in air deduced in Chapter III, which, if it were exact, would make $\sin 2\alpha = \frac{g}{V^2}(X + \frac{2}{3}kX^2)$.

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ballistic coefficient, which are best determined by direct experiment.

128. The Determination of C and j from the Experimental Ranging.—If the entire range table, and not the sight graduations alone, is to be based upon the results of experimental ranging, we may proceed as follows:

Compute by the ballistic tables the values of α for the various reduced measured ranges, and compare them with the experimental angles of elevation (quadrant angle plus angle of position).

If the difference between each value of α and the corresponding value of θ is the same, or nearly the same, that difference is evidently the jump, and the assumed value of C needs no correction.

If, as is much more likely to occur, the differences vary considerably from one range to another, we have to determine what correction of the value of C, together with some constant value of j, will most nearly make all the differences vanish. By equating the values of $\frac{dX}{X}$ given by (83) and (91), we obtain the relation between equivalent changes of C and α in the form $d\alpha = \frac{n-1}{2n} \tan 2\alpha \frac{dC}{C}$; and we wish to account for the differences between α and θ by a constant difference (j) and a varying difference caused by the correction of C (dC). Evidently then we have only to form the series of equations, one for each mean measured range,

$$\alpha_1 - \theta_1 = j + \frac{n_1 - 1}{2n_1} \tan 2\alpha_1 \frac{dC}{C}$$

$$\alpha_2 - \theta_2 = j + \frac{n_2 - 1}{2n_2} \tan 2\alpha_2 \frac{dC}{C}$$

&c.

and from them to determine the most probable values of j and $\frac{dC}{C}$. After which, with the jump j and the corrected value of C ($C \pm dC$) thus determined, all the required data for the range table may be computed by the regular methods.

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*EXAMPLES XII.

(1) A 5-inch gun, w = 50 lbs., V = 2300 f. s., was ranged with the following mean results:

Quadrant Angle.	Mean Range.
0° 30′	1178 yds.
1° 30′	2315 "
2° 00′	2716 "
3° 00′	3455 "
5° 00′	4647 "
7° 00′	5564 "
10° 00′	6850 "
13° 00′	7825 "

The barometer was 30.00″, thermometer 60°; it was calm; the height of gun above water was 12.3 ft.; and the jump was supposed to be zero. Determine the angles of elevation for horizontal ranges equal to those measured.

(2) The corrected angles of elevation from the data of Ex. (1) are 0° 41′ 57″; 1° 36′ 05″; 2° 05′ 13″; 3° 04′ 07″; 5° 03′ 03″; 7° 02′ 34″; 10° 02′ 04″; 13° 01′ 48″. Compute the values of n and reduce the measured ranges for the difference of the density of the air from its standard density.

(3) The values of n from the data of Ex. (1) are 1.134, 1.322, 1.467, 1.694, 2.067, 2.395, 2.743, 3.074; and the ranges, after reduction for the density of the air, are 3539, 6964, 8175, 10,407, 14,007, 16,777, 20,660, and 23,606 feet. Supposing there had been a wind component of 10 f. s. with the flight during the ranging, what would the reduced ranges be after taking account of the wind?

(4) After correction for the wind component assumed in Ex. (3), the ranges are 3536, 6950, 8155, 10,375, 13,949, 16,696, 20,542, and 23,455 feet. Supposing that the mean initial velocity during the ranging had been 2310 f. s., how much less would the ranges be for the standard velocity of 2300 f. s.? (Take values of n given in Ex. (3).) 27, 48, 54, 64, 80, 91, 109, 121 ft.

*The first four examples are intended to be given out successively, each one being divided among eight students.

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(5) The 12-inch gun, at an elevation of 7° 30′ by quadrant, gave ranges 11,912, 11, 867 and 11,924 yds. with initial velocities respectively 2828 f. s., 2832 f. s., and 2828 f. s. The height of the gun above water was 20.7 ft.; the barometer was 30.19″, and the temperature 59°, and it was calm. Taking the mean range and quadrant elevation, reduce them to standard conditions. (Standard I. V. is 2800 f. s.). 7° 32′; 11,812 yds.

(6) The 12-inch gun, at a quadrant elevation of 7° 0′, as a mean of three rounds, gave 2827 f. s. I. V. and 11,560 yds. range. The height of gun above water was 20.7 ft., the temperature was 59°, the barometer was 30.19″, and it was calm. Reduce to standard conditions. (Standard I. V. is 2800 f. s.).
7° 02′; 11,480 yds.

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CHAPTER XIII.

THE ACCURACY AND PROBABILITY OF GUN FIRE.

129. The Mean Point of Impact.—If a great number of rounds are fired from a gun, under as nearly as possible the same circumstances, it will be found that the impacts (which we will suppose to be received upon a vertical screen perpendicular to the plane of fire) are grouped closely together around one point and are more and more widely separated as the distance from that point increases.

The point about which the impacts are most closely grouped is called the *mean point of impact,* and its coordinates with reference to any axes Z, Y,—say the lower and the left-hand edges of the vertical screen,—are simply the averages of the coordinates of all the impacts with reference to the same axes. In other words, the mean point of impact is the center of gravity of the impacts, regarding them as equal masses, and its coordinates are $\frac{\Sigma z}{n}$ and $\frac{\Sigma y}{n}$ where $z_1 y_1$, $z_2 y_2$, &c., are the coordinates of the n impacts.

The trajectory from the gun to the mean point of impact is called the *mean trajectory,* and the divergencies of the trajectories of particular shots from the mean trajectory have a multitude of independent causes, such as,—an initial angular deviation; variations of the muzzle velocity; variations of the form and weight of the projectiles; variations in the angular position of the gun when it is fired, and in its jump; variations in the force and direction of the wind; and variations in the drift due to rotation.

If the actual mean initial velocity is that for which the sights of the gun are graduated; if the range is exactly known and the sights are set accordingly; if the density of the air is standard; and if there is no wind and no motion of gun or target; then the mean point of impact will coincide with the point at which the line of sight is directed. These conditions, however, are never fulfilled, nor is it possible to exactly compensate for their non-fulfillment, and consequently the mean point of impact

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is never exactly at the center of the target. To bring it as near as possible to that point is the object of the *regulation of gun-fire.*

130. **The Mean Deviation.**—For the present we will not concern ourselves with the position of the mean point of impact relative to the target, but will consider only the distribution of the impacts about the mean point of impact.

The deviations of projectiles from their mean point of impact are closely analogous to what are called "accidental errors," —such, for example, as are made in the direct measurement of a magnitude of any kind,—and they obey the same laws. Small deviations are more frequent than large ones; positive and negative deviations are equally probable, and are therefore equally frequent, if the number of shots is great; very large deviations are not to be expected at all (if one occurs, it must be due to some avoidable mistake). We will designate a lateral deviation by Δ_z and a vertical deviation by Δ_y, and we will assume that they are completely independent of one another,—i. e. that any particular lateral deviation is just as likely to occur simultaneously with one vertical deviation as with another. We will also designate the *mean lateral deviation* by γ_z and the *mean vertical deviation* by γ_y, the mean deviation being the arithmetical mean of all the deviations in the stated direction regardless of sign.

131. **Probability.**—The *probability* of a future event is the numerical measure of our reasonable expectation of its happening. Thus, knowing no reason to the contrary, we assign an equal probability to the turning up of each of the six different faces of a die at any throw, and we say that the probability that an ace, for example, will turn up is measured by the fraction 1/6. This does not mean that we should expect an ace to turn up once and once only in every six times, but merely that in a great number of throws (n) one may reasonably expect very nearly $\frac{n}{6}$ aces to be thrown and that the greater n is the more likely it is that the result will agree with that expectation.

If an event may happen in a ways and may fail in b ways, each of the $a + b$ ways being equally likely to occur, the probability of its happening is $\frac{a}{a+b}$ and the probability of its failing is $\frac{b}{a+b}$, and the sum of these two fractions, unity, represents the certainty that the event will either happen or fail. Thus if

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the probability of an event happening is P, the probability of its not happening is 1 — P. For example, since of the 52 different cards which may be drawn from a pack, 13 are spades, the probability that a single card drawn from a pack will be a spade is $\frac{13}{52} = \frac{1}{4}$, while the chance that it will not be a spade is $1 - \frac{13}{52} = \frac{39}{52} = \frac{3}{4}$.

If the probability of one event happening is P and that of another, independent, event happening is Q, then the probability of both events happening is the product of P and Q. For example, the probability of a single card drawn from a pack being a face card is $\frac{12}{52}$, and the chance of its being a spade is $\frac{13}{52}$; therefore the chance of its being either king, queen, or knave of spades is $\frac{12}{52} \times \frac{13}{52} = \frac{3}{52}$.

132. When we consider the deviation of projectiles, we are no longer dealing with definite numbers of equally probable events, but with values which may be anything whatever between certain limits. We cannot assign any finite measure to the probability that a deviation shall have a definite value because the number of values it may have is unlimited; we can, however, measure the probability that a deviation will fall between certain limits, or that it will be greater, or less, than an assigned quantity.

Suppose, for example, that a very large number (n) of shots have been fired, and that, their lateral deviations having been measured, it is found that m of these deviations are between 2 feet and 3 feet (either plus or minus); then we say that the probability (in any future trial under similar circumstances) of a single shot having a lateral deviation between 2 and 3 feet is $\frac{m}{n}$. Or if, of the n impacts, q were less than 4 feet to one side or the other of the mean point of impact, we should say that the probability of the lateral deviation of any single shot being less than 4 feet is $\frac{q}{n}$. The following actual case will serve as an illustration, although n is not so large as it should be.

On December 17, 1880, at Krupp's proving-ground, at Meppen, 50 shots were fired from a 12 cm. siege gun at 5° elevation, giving a mean range of 2894.3 meters. The points of fall

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being marked on the ground, their distances from assumed axes were measured, and the mean point of impact thus determined. Measuring the lateral deviations from this mean point of impact to the right, 14 were between 0 and 1 meter, 8 between 1 and 2 meters, and 2 between 2 and 3 meters; to the left 13 were between 0 and 1 meter, 8 between 1 and 2 meters, and 5 between 2 and 3 meters; and the mean deviation was 1.07 meters. Taking horizontal and vertical axes through the mean point of impact; laying off equal spaces to left and right of the origin, each representing one meter; and constructing on each space a rectangle whose height represents, on any convenient scale, the number of shots whose lateral deviation was within the limits corresponding to the space, we obtain Fig. 20.

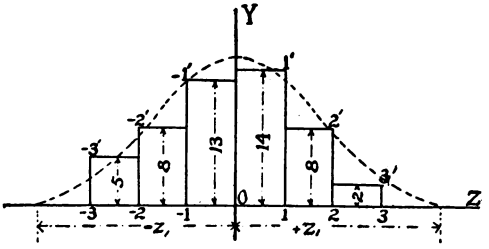

Fig. 20.

It will be seen that the distribution of the deviations is fairly symmetrical to the Y axis, there being 26 to the left and 24 to the right; also that the maximum does not exceed three times the mean deviation; also that the area of each rectangle divided by the whole area of the figure is the measure of the probability (as defined) that any single deviation will fall within the limits represented by its base. Thus area O 11′ = 14, divided by total area = 50, is the probability = $\frac{14}{50}$ of any single deviation being between 0 and + 1m.; area 11′ — 1 — 1′ = 27, divided by total area = 50, is the probability = $\frac{27}{50}$ that any single deviation will be between + 1 m. and — 1 m.; and the total area = 50, divided by itself, is the probability = $\frac{50}{50}$ = certainty that no deviation will exceed 3 m.

If now the number of shots is increased, while the width

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of the horizontal spaces is diminished in the same proportion, the area of each rectangle divided by the whole area of the figure will continue to measure (with increasing accuracy) the probability of a deviation falling within the limits represented by its base. At the limit, when the number of shots is infinite and the width of the horizontal spaces has been reduced to the infinitesimal dz, the height (y) of each rectangle will still be finite; the upper contour of the figure will become a curve approximately like that shown as a dotted line; the area of each rectangle (now become the elementary area ydz) divided by the whole area under the curve $\left(\int_{-z_1}^{z_1} y dz\right)$ will still measure the probability that a deviation will fall within the limits represented by its base (between z and $z + dz$); and the area between any two ordinates of the curve $\left(\int_b^a y dz\right)$ divided by the whole area will still measure the probability that a deviation will be between a and b.

133. **The Probability Curve.**—The curve just described is the *probability curve* for the lateral deviations of the projectiles of the particular gun considered, and while the probability curve for the vertical deviations, or that for either lateral or vertical deviations in the case of any other gun, would differ somewhat from this particular curve, all would present the following general features: (1) Since plus and minus deviations are equally likely to occur, the curve must be symmetrical to the right and left of the origin, which is the mean point of impact. (2) Since the deviations are made up of elemental deviations which, as they may have either direction, tend to cancel one another, small deviations are more frequent than large ones, so that the maximum ordinate occurs at the origin. (3) Since large deviations (which can only result when most of the elemental deviations have the same direction and their greatest magnitude) are rare, and deviations beyond a certain limit do not occur, the curve must rapidly approach the horizontal axis both to right and left, so that the ordinate (which can never be negative) practically vanishes at a certain distance from the origin.

If $y = \phi(z)$ is the equation of the probability curve, the general features just stated require, (1) that $\phi(z)$ shall be an even function, that is, a function of z^2, (2) that $\phi(0)$ shall be its maximum value, (3) that it shall be a decreasing function of z^2 and

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shall practically vanish when z is large. Since it is impracticable to select the function ϕ so that $\phi(z)$ shall be constantly equal to zero when z exceeds a certain limit, the last condition requires that the curve shall have the axis of z for an asymptote; in other words, we must have $\phi(\pm\infty) = 0$.

The foregoing characteristics being thus established, and taking as its basis the axiom that the arithmetical mean of the observed values (made under similar circumstances and with equal care) of any quantity is its most probable value, the theory of accidental errors deduces as the equation to the probability curve.

$$y = \frac{1}{\pi\gamma}\epsilon^{-\frac{z^2}{\pi\gamma^2}}$$

In which γ is the mean error, or in our case the mean deviation, $\pi = 3.14159$, and $\epsilon = 2.71828$, and the factor $\frac{1}{\pi\gamma}$ has been intro-

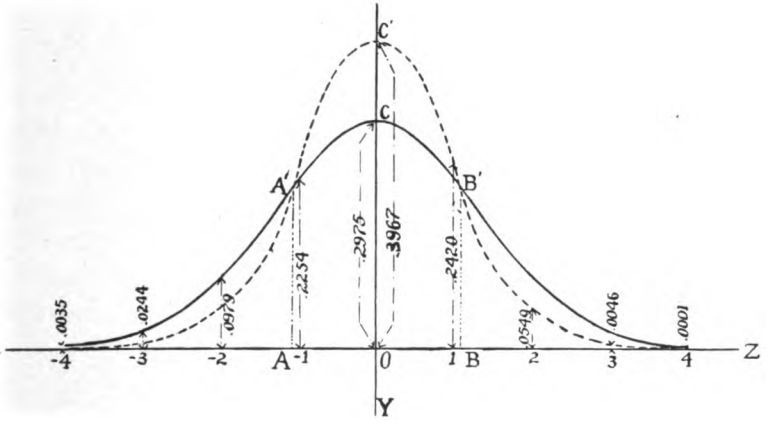

Fig. 21.

duced to make the whole area under the curve equal to unity $\left(\int_{-\infty}^{+\infty} \epsilon^{-\frac{z^2}{\pi\gamma^2}}\, dz = \pi\gamma\right)$, thus obviating the necessity of dividing a partial area by the whole area (in the manner described above) whenever a probability is to be computed.

Figure 21 represents the probability curve for the Krupp 12 cm. siege gun, taking its mean error to be 1.07 meters, as given

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by the 50 shots referred to above; also, in dotted lines, for comparison, the probability curve for a gun whose mean error is ¾ that of the 12 cm. gun. In both curves the ordinates are exaggerated ten times as compared with the abscissæ.

The maximum ordinate, being the value of y when $z = 0$, is inversely proportional to the mean deviation $\left(y\Big]_{z=0} = \frac{1}{\pi\gamma}\right)$: the probability that a deviation will be less than $OB = OA$ is the numerical value of the area AA'CB'B in the one case and AA'C'B'B in the other case; the probability that a deviation will exceed $OB = OA$ is the area under that part of the curve which is to the left of AA' and to the right of BB': the whole area under each curve has the numerical value one. It will be seen how very small is the probability of any deviation exceeding three times the mean deviation.

135. The probability (P) that the deviation of any single shot will be numerically less than a given quantity (a) being measured by the area between the ordinates of the probability curve at $z = +a$ and $z = -a$, and that curve being symmetrical to the axis of Y, we have

$$P = \frac{2}{\pi\gamma}\int_0^a \epsilon^{-\frac{z^2}{\pi\gamma^2}}\,dz.$$

The following table gives the values of P, calculated from this equation, but arranged for convenient use with the ratio $\frac{a}{\gamma}$ as the argument. Knowing the mean lateral deviation (γ_z) of a gun, to find the probability of a shot striking within the distance a to right or left of the mean point of impact, it is only necessary to take from the table the value of P which corresponds to the argument $\frac{a}{\gamma_z}$. And so, too, the probability of a shot striking within the distance a above or below the mean point of impact is the value of P which corresponds to the argument $\frac{a}{\gamma_y}$, γ_y being the mean vertical deviation.

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Probability of a deviation less than a in terms of the ratio $\frac{a}{\gamma}$.

$\frac{a}{\gamma}$	P	$\frac{a}{\gamma}$	P	$\frac{a}{\gamma}$	P	$\frac{a}{\gamma}$	P
0.1	.064	1.1	.620	2.1	.906	3.1	.987
0.2	.127	1.2	.662	2.2	.921	3.2	.990
0 3	.189	1.3	.700	2.3	.934	3.3	.992
0.4	.250	1.4	.735	2.4	.945	3.4	.994
0.5	.310	1.5	.768	2.5	.954	3.5	.995
0.6	.368	1.6	.798	2.6	.962	3.6	.996
0.7	.424	1.7	.825	2.7	.969	3.7	.997
0.8	.477	1.8	.849	2.8	.974	3.8	.998
0.9	.527	1.9	.870	2.9	.979	3.9	.998
1.0	.575	2.0	.889	3.0	.983	4.0	.999

136. As an illustration of the use of the table, we will find the probability of the deviation not exceeding 1 meter and 2 meters in the case of a gun whose mean lateral deviation is 1.07 meters, and will compare our results with those given by the actual firing of 50 shots from the Krupp 12 cm. gun. Taking $a = 1$ meter, we have $\frac{a}{\gamma} = .935$, and $P = .544$; the probability that the lateral deviation of a single shot will not exceed one meter is .544; therefore of 50 shots 27 should fall within one meter on either side of the mean point of impact; actually 27 did so fall. Taking $a = 2$ meters, we have $\frac{a}{\gamma} = 1.87$, and $P = .864$; the probability that the lateral deviation of a single shot will not exceed 2 meters is .864; therefore of 50 shots 43 should fall within 2 meters on either side of the mean point of impact; actually 43 did so fall.

137. If P is the probability that the deviation of any single shot will not be greater than a, then evidently 100 P will be the probable number of shots out of 100 which will fall within the limits $\pm a$; in other words, 100 P is the percentage of hits to be expected upon a band 2a wide with its center at the mean point of impact. Thus we see, from the table, that the half width of the band which will probably receive 25% of the shot is 0.4γ, while the half width of the band which will probably receive 50% of the shot is $0.846\,\gamma$. These facts are usually expressed by saying that the widths of the 25% rectangle and of the 50% rectangle are respectively 0.80 and 1.69 times the mean error.

The half width of the 50% rectangle is known as the *probable error,* or in our case the *probable deviation,* since it is

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the error, or deviation, which is just as likely to be exceeded as it is not to be exceeded.

If we wish to find the probability of hitting an area whose width is 2b and whose height is 2h, since the lateral and vertical deviations are independent of one another, the probability is the product of the two values of P taken from the table with the arguments $\frac{b}{\gamma_z}$ and $\frac{h}{\gamma_y}$, where γ_z and γ_y are the mean lateral and the mean vertical deviations respectively. Thus, supposing γ_z to be 4 feet and γ_y to be 5 feet, the probability of hitting with a single shot a 20-foot square with center at the mean point of impact is $P_1P_2 = .954 \times .889 = .848$, $P_1 = .954$ being the value of P for $\frac{b}{\gamma_z} = \frac{10}{4} = 2.5$ and $P_2 = .889$ being the value of P for $\frac{h}{\gamma_y} = \frac{10}{5} = 2$.

EXAMPLES XIII.

(1) The coordinates (z, y) of 10 hits made by a 6 pdr. gun on a vertical target at 2000 yds. range, axes at center of target, were as follows, in feet: (— 10, + 13), (+ 11, + 9), (+ 4, — 2), (— 1, + 1), (— 4, + 2), (+ 2, + 1), (— 1, — 2), (0, — 4), (— 1, — 3), (— 4, — 4). Find the mean point of impact and the mean vertical and lateral deviations.

$$z_0 = -0.4;\ y_0 = +1.1;\ \gamma_y = 4.14;\ \gamma_z = 3.72.$$

(2) The coordinates of 8 hits made by a 28 cm. gun on a vertical target at 4019 meters range, axes at center of target, were as follows, in centimeters: (— 80, — 90), (— 10, + 210), (+ 30, — 70), (— 70, + 355), (+ 30, + 40), (— 220, — 150), (— 40, + 40), (— 65, + 90). Find the mean point of impact and the mean vertical and lateral deviations.

$$z_0 = -53;\ y_0 = +53;\ \gamma_y = 123.9;\ \gamma_z = 55.7.$$

(3) The following ranges and lateral deflections from the plane of fire, in meters, were given by 10 shots from a 28 cm. gun at 8° 30′ elevation:

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Range.	Deflection, left.	Range.	Deflection, left.
6285 m	18 m.	6204 m.	16 m.
6228	21	6141	17
6187	15	6200	19
6187	12	6256	15
6192	17	6205	18

Find the mean point of impact, the mean lateral deviation, and the mean deviation in range.

Mean range 6208.5 m.; $\gamma_x = 28.7$ m.
Mean lateral deflection 16.8 m.; $\gamma_z = 1.8$ m.

(4) A and B shoot alternately at a mark. If A can hit once in n trials and B once in n — 1 trials, show that their chances are equal for making the first hit. What are the odds in favor of B after A has missed the first shot? n to n — 2.

(5) What is the probability of throwing an ace with a single die in two trials? $\frac{11}{36}$.

(6) What is the probability of throwing two aces, and no more, in a single throw of five dice? What is the probability of throwing at least two aces? $\frac{1250}{6^5} = .1608$; $\frac{1526}{6^5} = .1962$.

(7) Taking the mean vertical error given from Ex. (1), and supposing the mean point of impact to be at the center of a vertical target, what would be the percentages of hits on targets of unlimited width and of heights respectively 8 ft., 12 ft., 16 ft., 20 ft., and 24 ft.? 55.9%; 75.1%; 87.6%; 94.6%; 97.9%.

(8) Taking the mean errors given by Ex. (1), what percentage of shot would enter a gun port 4 feet square, supposing the mean point of impact to be at the center of the port? What would be the percentage if the port was 3 ft. high by 5 ft. wide? 9.9%; 9.2%.

(9) What would be the probability of a single shot from the 28 cm. gun of Ex. (2) hitting a turret 2 meters high and 8 meters in diameter at the range for which the mean errors are given, supposing the fire to be exactly regulated? .48.

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(10) If a zone of a certain width receives 20% of hits, how many times as wide is the zone which receives 80% of hits?

5.05 times.

(11) At Bucharest, in 1886, 94 shots were fired from a Krupp 21 cm. rifled mortar at a Gruson turret, distant 2510 meters, without hitting it. The mean deviations were 33.27 meters in range and 9.90 meters laterally, and the mean point of impact practically coincided with the center of the turret. What was the probability of hitting, supposing the target to have been a 6 meter square (it was really a circle of 6 m. diameter)? .011.

(12) How many of the 94 shots of Ex. (11) would probably have struck a rectangle 80 m. by 16 m. with the longer axis in the plane of fire? 31.8%.

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CHAPTER XIV.

THE ACCURACY AND PROBABILITY OF GUN-FIRE, CONTINUED.—THE MEAN ERRORS OF GUNS.

138. We have thus far only considered the chances of hitting when the mean point of impact is at the center of the target, but this is very far from being an attainable condition in the service use of guns, especially of naval guns. Even with a stationary target, at a known range, it is difficult to so regulate the fire as to bring the mean point of impact, after a few trial rounds, close to the center of the target, and when the target is moving, with a speed and in a direction both only estimated, when the range is but approximately known, and when the gun itself is in motion, the most expert regulation of the fire by observation of the points of fall can do no more than to keep the mean point of impact in the neighborhood of the object attacked.

Let O be the mean point of impact, and ABCD the target (Figure 22) at any moment, and let the coordinates of the center of ABCD, with reference to vertical and horizontal axes through O, be y_0 and z_0; also let the mean vertical and lateral deviations be γ_y and γ_z respectively and let the dimensions of the target be 2h and 2b. Then the probability of a shot falling between the vertical lines Cc and C'c' is the tabular value of P for the argument $\frac{z_0+b}{\gamma_z}$ (which we will call $P_z(z_0+b)$), and the probability of a shot falling between Dd and D'd' is the tabular value of P for the argument $\frac{z_0-b}{\gamma_z}$ ($P_z(z_0-b)$). Therefore the probability of a shot falling between Cc and Dd is one-half the difference of the two preceding probabilities, or $\frac{1}{2}[P_z(z_0+b) - P_z(z_0-b)]$. Similarly the probability of a shot falling between the horizontal lines C'C and B'B is $\frac{1}{2}[P_y(y_0+h) - P_y(y_0-h)]$. Hence the probability of hitting ABCD is the product of these two quantities, or $\frac{1}{4}[P_z(z_0+b) - P_z(z_0-b)]\ [P_y(y_0+h) - P_y(y_0-h)]$.

139. To illustrate, suppose we wish to find the probable percentage of hits on a gun-port 4 feet square, if the mean point of

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impact is 3 feet to one side and 4 feet below the center of the port, the value of γ_z being 3.72 feet and that of γ_y being 4.14 feet. Here we have $P_z(z_0+b) = P(5) = .717$; $P_z(z_0-b) = P_z(1) = .170$; $P_y(y_0+h) = P_y(6) = .751$; $P_y(y_0-h) = P_y(2) = .298$. Therefore the probability of hitting the port is $\frac{1}{4}(.717-.170)(.751-.298) = \frac{1}{4} \times .547 \times .453 = .062$, and the percentage of hits would be 6.2. Under the same circumstances, but with the mean point of impact at the center of the port, the percentage of hits would be 9.9.

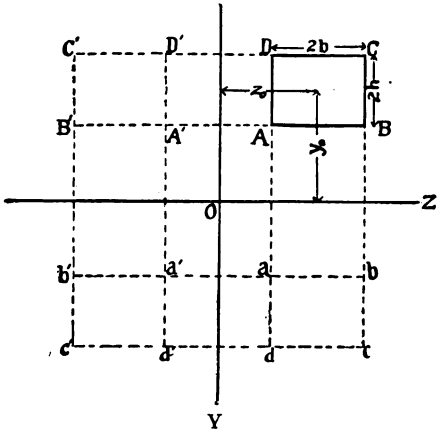

Fig. 22.

140. Evidently the less the mean errors of a gun the more important it is to accurately regulate its fire; if the distance of the mean point of impact from the target is more than three times the mean error of the gun there will be practically no hits at all.

To illustrate the preceding statement, we will take the case of a 6-inch gun firing at a heavy gun position, 25 feet high by 32 feet in diameter and 3000 yards distant. Supposing the mean vertical and lateral deviations each to be 10 feet; if the mean point of impact coincides with the center of the target, the probable percentage of hits will be found to be 54.3; but if the sights are set for a range 10% more or less than the true distance the mean point of impact will be raised, or lowered, about 43 feet, and the percentage of hits will be reduced to 0.7. If, on the other hand, the mean errors of the gun were each 20 feet, or double what was first assumed, while the percentage of hits with

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perfect regulation of the fire would be reduced to 18.2, that with sight setting for a range 10% in error would be increased to 4.7. Thus we see that a gun may be severely handicapped by its own accuracy if the range is not known within 10%.

141. If we know the percentage of hits at a given range on a target of given size, we can make a rough estimate of the mean errors of the gun by assuming that the mean point of impact was at the center of the target, and the greater the number of rounds fired the more nearly correct will the determination probably be. For example, the eighty 6-inch guns of certain British ships, firing separately, made 295 hits out of 650 rounds fired, 45.4% of hits, on a target 15 feet high by 20 feet wide at a mean range of 1500 yards. Here we have given that the product of $P\left(\frac{10}{\gamma}\right)$ by $P\left(\frac{7.5}{\gamma}\right)$ is .454, and by a process of trial and error we readily find from the table that γ is slightly less than 7 feet, since $P\left(\frac{10}{7}\right)$ by $P\left(\frac{7.5}{7}\right) = .745 \times .581 = .432$.

The number of rounds necessary to make at least one hit is determined as follows: Let p be the probability of hitting with a single shot; then $1 - p$ is the probability of a single shot missing; and $(1-p)^n$ is the probability that all of n shots will miss; hence the probability of hitting at least once with n shots is $P = 1 - (1-p)^n$. Solving this equation for n, we have $\log(1-P) = n \log(1-p)$; $n = \frac{\log(1-P)}{\log(1-p)}$; and by giving P a value near unity we can find the value of n which will make one hit as nearly certain as we wish. As an illustration, taking a case in which 94 shots were fired from a mortar at a Gruson turret and in which the calculated probability of a hit with a single shot was .011, let us see how many rounds should have been fired to make the probability of at least one hit .95. In that case $p = .011$, and so we have $n = \frac{\log(1-.95)}{\log(1-.011)} = \frac{8.69897\ (-10)}{9.99520\ (-10)} = \frac{-1.30103}{-0.00480} = 271$; so that 271 shots must be fired to make the odds 19 to 1 that there will be at least one hit. The probability of at least one hit with the 94 shots actually fired was $P = 1 - (1-p)^{94} = 1 - .354 = .646$.

142. The Mean Errors of Guns on Steady Platforms.—The

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deviations of the projectiles fired from a gun on a steady platform, the mean of which we will call the mean error of the gun, lateral or vertical as the case may be, are principally caused by (1) errors of the gun pointer in sighting the gun, (2) an initial angular deviation of the projectile, (3) variations of the initial velocity, and (4) differences between the projectiles.

With open sights the most expert gun pointers make a considerable angular error, which varies from round to round, when they point a gun by directing its line of sight at a mark. With telescope sights this error is greatly lessened, but still exists. There is also always an error in setting the sights (we suppose the range to be unchanged, but the sights to be reset for each round). The *mean angular error of sighting* can only be estimated, but it is unlikely that it is less than one minute, even with the telescope sight, under ordinary conditions of rapid firing; the resulting mean deviation at 2000 yards range would then be about 21 inches, and may be taken to occur indifferently in all directions, though it is somewhat greater vertically than laterally. Such a deviation is equivalent to a mean vertical deviation of $\frac{21}{\sqrt{2}} = 15$ inches together with a mean lateral deviation of the same value.

The *initial angular deviation* results from the projectile not leaving the gun in the exact line of the latter's axis. This deviation, which occurs indifferently in all directions, was quite large with smooth-bore guns and with some of the early rifles, but with modern guns, using projectiles rotated by forced bands, its mean value is probably not more than half a minute of arc. The mean deviation of the projectiles at 2000 yards' range would be about 10.5 inches by reason of this cause, and this is equivalent to a mean vertical deviation of $\frac{10.5}{\sqrt{2}} = 7.5$ inches together with a mean lateral deviation of the same value.

With all powders the muzzle velocity varies from round to round, no matter what care be taken to insure uniformity of the charges. With the gun-cotton smokeless powder now used in the U. S. Navy, if the charges have been made up with proper care, and the projectiles are all of the same weight, the average difference between the velocities given by successive rounds and the mean velocity of all the rounds fired on any one occasion will

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probably not exceed 10 f. s. That is, if a large number of rounds are fired, and the average velocity is, say 2400 f. s., then probably no one velocity will differ more than 30 f. s. from the average and somewhat more than half the velocities will be within 10 f. s. of the average. Such a variation will cause a mean vertical deviation of the projectiles at 2000 yards range of 11 or 12 inches, for guns whose average initial velocity is 2400 f. s., and of 7 or 8 inches for guns having 2800 f. s. muzzle velocity.

The projectiles of any gun differ among themselves in shape and smoothness, but, when they all have the same form of head and the various kinds are not of greatly different lengths, the resulting deviations are not so important as those caused by the variations in their weights. If we take the mean variation in weight to be 1% of the standard weight, the resulting mean vertical deviation at 2000 yards' range will be about 9 inches.

Neglecting the other numerous, but lesser, causes of deviation, and assuming that those described are independent of one another, we find the mean vertical and the mean lateral deviation which results from their combined action by taking the square root of the sum of the squares of the corresponding partial mean deviations.* And thus we have $\gamma_y = (15^2 + 7.5^2 + 9.5^2 + 9^2)^{\frac{1}{2}} =$ 21¼ inches, and $\gamma_z = (15^2 + 7.5^2)^{\frac{1}{2}} =$ 16¾ inches.

143. Of course the only correct way of determining the mean errors of a given gun is by actually firing a large number of rounds at a target and measuring the deviations, but the foregoing estimate will serve to give an idea of the errors to be expected in the case of modern guns of medium and large caliber. That the errors are very small under the favorable circumstances thus far considered is illustrated by Figure 23, which represents a target made at Meppen on June 1, 1882, with a 28-cm. gun, the distance of the target from the gun being 2026 meters (2215 yards). The dotted cross is the mean point of impact, whose coordinates, referred to horizontal and vertical axes at the center of the target, are $z = 32.4$ inches, and $y = -11.6$ inches. The mean lateral deviation is 9.5 inches, and the mean vertical deviation is 11.6 inches.

The fact that these actual deviations are considerably less than

*It is demonstrated in the Calculus of Probabilities that the mean value of the total error resulting from any number of independent causes is the square root of the sum of the squares of the mean partial errors.

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our estimated ones may be plausibly ascribed to the fact that in proving ground firings greater care is taken in pointing the gun than is usually practicable under service conditions.

144. The Effect of Motion of the Gun Platform and Target.— The three angular motions of a ship's deck, caused by rolling, pitching, and yawing, greatly increase the mean errors of naval guns, but their effects depend so much upon the skill of the gun pointer, as well as upon the state of the sea and the characteristics of the particular ship and gun mounting, that only the roughest estimate of their values can be made.

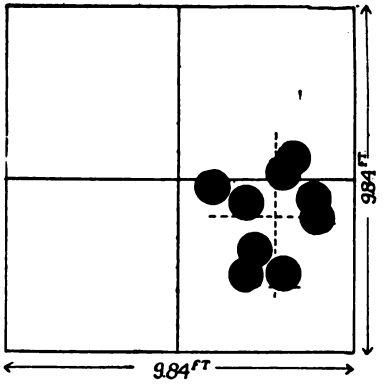

Fig. 2[illegible].

Most naval guns are mounted in broadside and only train from bow to quarter, and even guns mounted on the middle line are likely to be used most on the broadside; thus the roll, which is the greatest and most rapid of a ship's angular motions, has its largest component in the plane of fire, and acts principally to increase the vertical deviations. The principal effect of pitching, on the other hand, is to increase the lateral deviations by causing the plane of the sights to be more or less inclined, now to one and now to the other side of the plane of fire. Motion in azimuth, mostly due to unsteady steering, affects the lateral deviations only.

If the target is in motion, the gun pointer must, of course, estimate its speed and direction in order to direct his fire at the point where the target will be when the projectile strikes, and his corrections must always vary in their accuracy from round to round, thus increasing both the lateral and the vertical deviations.

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Moreover, the changing direction of the target will cause the angle between the direction of the wind and the plane of fire to vary, thus necessitating a variable allowance for wind effect and again increasing the deviations of the projectiles.

145. Taking everything into account, probably a fair estimate of the mean vertical deviation of modern naval guns of medium and large caliber at 2000 yards range, with skilled gun pointers, and under average conditions, would be 5 feet. The mean lateral deviation, which for guns on steady platforms is from two-thirds to three-fourths the mean vertical deviation, may be taken to be the same as the mean vertical deviation in the case of naval guns without any great error.

Both vertical and lateral deviations may be taken to be proportional to the range, at least up to 4000 or 5000 yards range, though the former really increases somewhat more rapidly than the range.

Although the targets of naval guns are vertical, their fire must generally be regulated by observation of the points of fall on a horizontal plane. The lateral deviations are practically the same whether measured on the vertical plane (perpendicular to the plane of fire) or on the horizontal plane, but the deviations in range differ very greatly from the vertical deviations, the ratio between them being the cotangent of the angle of fall.

146. The Mean Deviation in Range.—Since the mean deviation in range (γ_x) is related to the mean vertical deviation (γ_y) by the formula $\gamma_x = \gamma_y \cot \omega$, and since the angle of fall increases (and its cotangent correspondingly decreases) with increase of range at about the same rate as the mean vertical deviation, it will be seen that the mean deviation in range remains nearly the same for widely different ranges. Thus, for example, while the estimated mean vertical deviation of the 12-inch gun of 2800 f. s. I. V. increases from 2.5 feet at 1000 yards range to 10.5 feet at 4000 yards range, the corresponding mean deviation in range only changes from 119 yards to 104 yards; and while in the case of smaller guns the mean deviation in range decreases more rapidly, still the change is always very much less proportionately than the change in range itself.

147. The principal use of knowledge of the mean deviation in range is in the regulation of gun-fire by observation of the points of fall.

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Suppose the axis of Z in Figure 24 represents the water line of the target, the axis of X being the trace of the plane of fire; and let the distances from Z to the dotted lines aa, a'a', bb, b'b', etc., represent once, twice, etc., the mean deviation in range (γ_x) of the gun. Then if the mean point of impact is on Z, half of all the shots will fall short; if it is on aa, the percentage of shots which fall short will be increased by the number which fall between Z and aa, or, from the table of probabilities, it will be $50 + \frac{57.5}{2} = 79\%$; if the mean range is still further short, so that the mean point of impact falls on bb, the percentage of shorts will be $50 + \frac{88.9}{2} = 94\%$; and, finally, if the mean point of impact is three or more times the mean deviation short, then practically

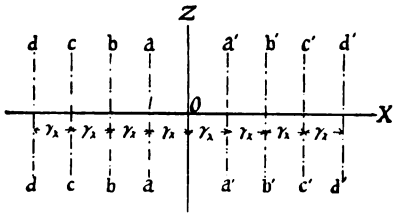

Fig. 24.

all the shot will fall short. The same reasoning shows that if no shot strike short of Z, the mean point of impact is three or more times the mean deviation in range beyond Z; if about 6% strike short, the mean point of impact is about $2\gamma_x$ beyond Z; and if about 21% are short, it is about γ_x beyond Z.

Thus by observation of the percentage of shot which strike short it is possible to determine with some accuracy how much the sight bars should be raised or lowered to bring the mean point of impact on the target.

148. How closely the calculated percentages of hits agree with the results of experiment when the number of shots fired is considerable, is illustrated by the following table taken from Hélie's well-known Traité de Balistique. It represents about 500 shots fired at Gavre from the 16.5-cm. rifle at various angles of elevation.

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Probability that the lateral deviation will not exceed

	$\frac{\gamma z}{4}$	$\frac{\gamma z}{2}$	γz	$2\gamma z$	$3\gamma z$
By table	0.158	0.310	0.575	0.889	0·983
By experience	0.176	0.300	0.592	0.885	0.988

EXAMPLES XIV.

(1) Supposing a row of gun ports, each 4 ft. high by 6 ft. wide, are spaced 18 ft. between centers; show by comparing the percentages of shot which would enter a port that it would be better to aim a gun whose mean vertical and lateral errors are each 5 ft. at the center of a port than half way between two ports.

9.2% to 7.1%.

(2) Compare the percentages of hits on ports in the two cases of Ex. (1) if the mean errors of the gun were 7.5 ft. instead of 5 ft. 4.2% to 5.4%.

(3) The 12-inch guns of a certain ship made 68% of hits on a target 15 ft. high by 20 ft. wide at 1700 yards range. Supposing the mean vertical and lateral deviations to have been equal, what was their approximate value? Supposing through ignorance of the range the sights had been set for 1870 yds., thus raising the mean point of impact 9 feet, what would the percentage of hits have been? 5 ft.; 35.7%.

(4) The mean errors, laterally and in range, of a rifled mortar are 3.5 yds. and 53 yds. respectively, at a mean range of 3357 yds. What is the chance of hitting a ship's deck (taking its equivalent area to be a rectangle 300 ft. by 60 ft.) when she is end on, (1) if the mean point of impact is at one corner of the rectangle; (2) if it is at the center of the rectangle?

.217; .535.

(5) What are the chances of hitting under the same circumstances as in Ex. (4), excepting that the ship is broadside on?

.059; .120.

(6) At a range at which the mean vertical error of the guns equals the freeboard of an enemy, what is the ratio of the respective probabilities of hitting her when you aim at her water line and when you aim at her middle height, suppose the fire to

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be accurately regulated in each case? What is the same ratio at a range at which the mean vertical error is only half the freeboard? .288 to .310; .445 to .575.

(7) A torpedo boat steaming directly for a ship at 24 knots is discovered and fire is opened on her at 1500 yds. range. If the probability of a single 3-inch shot striking her is .02, and there are eight 3-inch guns each firing 12 rounds a minute at her, what is the chance that she will be struck at least once before she is within 500 yds.? What is the chance of her being hit at least twice? .911; .694.

(8) At 4000 yds. a ship with 30 feet freeboard gives a danger space of 90 yds. for the 3-inch gun (I. V. = 2800 f. s.) and the mean error in range of the same gun at 4000 yds. is 30 yds. (estimated). How closely must the range of such a ship be known to make the probable percentage of hits as great as 0.5%, supposing the guns to be pointed at the middle of the freeboard? 141 yds.

(9) The mean error in range of the 12-inch gun (I. V. = 2800 f. s.) at 4000 yds. range is 100 yds. (estimated), and its danger space for 30 feet freeboard is 300 yards. How closely must the range be known to make the probable percentage of hits as great as 0.5%, supposing the guns to be pointed at the middle height of the freeboard? 470 yds.

(10) Fire is opened with eight 3-inch guns on a torpedo boat coming head on when she is at 1500 yds. range. She covers 100 yds. every 7½ seconds, and each gun fires once every 7½ seconds. The mean lateral and vertical deviations are each 6 feet, and the target offered is 6 ft. high by 15 ft. wide. If an error of 100 yds. in the sight-setting displaces the mean point of impact 3 ft. vertically, and the sights are all kept set for 1000 yds. range, what is the probable number of hits while the boat advances to 500 yds. range? 10 +.

(11) The turrets of a monitor steaming obliquely to the line of fire present a vertical target consisting of two rectangles, each 24 ft. wide by 12 ft. high, and 36 ft. from center to center. If the mean errors of a gun are 12 yds. laterally and 8 yds. vertically, would it be better to aim at a turret or half way between them? 1st case, P = .057; 2d case, P = .061.

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(12) A gun has 30 shell, one of which, if landed in a Moncrieff pit, would silence it. The pit is 10 yds. in diameter, and the probability of hitting it with the gun in question is .05. What would be the probability of silencing the pit, using all the ammunition? P = .785.

(13) The 12-inch guns of a ship made 68% of hits on a target 15 ft. high by 20 ft. wide at 1700 yds. range. What was the probable value of the mean deviation, vertical and lateral? Supposing the mean deviation to be proportional to the range, what percentage of hits would the same guns make on the same target at 3400 and at 5100 yards? 5 ft.; 25.9%; 12.6%.

(14) If the probability of hitting a target with a single shot is .05, what will the probability be of making at least two hits with 50 shots? .721.

(15) What is the greatest value of the mean deviation of a gun consistent with a probability equal to .90, of its making at least one hit in a hundred shots on a gun port 2 ft. wide by 4 ft. high? 5.95 ft.

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TABLE I.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
3600	0.0	54.5	0.00	1.72	0.03138	27	0.000	15
3580	54.5	54.7	1.72	1.74	0.03165	28	0.015	15
3560	109.2	54.9	3.46	1.76	0.03193	28	0.030	16
3540	164.1	55.1	5.22	1.78	0.03221	28	0.046	16
3520	219.2	55.2	7.00	1.80	0.03249	29	0.062	15
3500	274.4	55.4	8.80	1.82	0.03278	30	0.077	16
3480	329.8	55.5	10.62	1.85	0.03308	29	0.093	16
3460	385.3	55.8	12.47	1.87	0.03337	30	0.109	16
3440	441.1	55.9	14.34	1.89	0.03367	31	0.125	17
3420	497.0	56.1	16.23	1.92	0.03398	31	0.142	16
3400	553.1	56.3	18.15	1.93	0.03429	32	0.158	17
3380	609.4	56.4	20.08	1.96	0.03461	32	0.175	17
3360	665.8	56.7	22.04	1.99	0.03493	32	0.192	16
3340	722.5	56.8	24.03	2.02	0.03525	33	0.208	17
3320	779.3	57.1	26.05	2.04	0.03558	33	0.225	18
3300	836.4	57.2	28.09	2.06	0.03591	34	0.243	17
3280	893.6	57.4	30.15	2.09	0.03625	35	0.260	18
3260	951.0	57.6	32.24	2.12	0.03660	35	0.278	17
3240	1008.6	57.8	34.36	2.15	0.03695	36	0.295	18
3220	1066.4	58.0	36.51	2.17	0.03731	36	0.313	18
3200	1124.4	58.2	38.68	2.20	0.03767	37	0.331	18
3180	1182.6	58.4	40.88	2.24	0.03804	37	0.349	19
3160	1241.0	58.6	43.12	2.26	0.03841	38	0.368	18
3140	1299.6	58.8	45.38	2.29	0.03879	39	0.386	19
3120	1358.4	59.0	47.67	2.32	0.03918	39	0.405	19
3100	1417.4	59.2	49.99	2.36	0.03957	40	0.424	19
3080	1476.6	59.5	52.35	2.39	0.03997	40	0.443	20
3060	1536.1	59.6	54.74	2.42	0.04037	42	0.463	19
3040	1595.7	59.9	57.16	2.45	0.04079	42	0.482	20
3020	1655.6	60.1	59.61	2.49	0.04121	42	0.502	20
3000	1715.7	60.3	62.10	2.52	0.04163	44	0.522	20
2980	1776.0	60.5	64.62	2.56	0.04207	44	0.542	21
2960	1836.5	60.7	67.18	2.60	0.04251	45	0.563	20
2940	1897.2	61.0	69.78	2.64	0.04296	45	0.583	21
2920	1958.2	61.2	72.42	2.67	0.04341	47	0.604	21
2900	2019.4	61.5	75.09	2.71	0.04388	47	0.625	21
2880	2080.9	61.7	77.80	2.75	0.04435	48	0.646	22
2860	2142.6	61.9	80.55	2.79	0.04483	49	0.668	22
2840	2204.5	62.2	83.34	2.83	0.04532	50	0.690	22
2820	2266.7	62.4	86.17	2.88	0.04582	51	0.712	22
2800	2329.1	62.6	89.05	2.92	0.04633	52	0.734	22
2780	2391.7	62.9	91.97	2.96	0.04685	53	0.756	23
2760	2454.6	63.1	94.93	3.01	0.04738	53	0.779	23
2740	2517.7	63.4	97.94	3 05	0.04791	55	0.802	23
2720	2581.1	63.7	100.99	3.10	0.04846	56	0.825	24

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
2700	2644.8	63.9	104.09	3.16	0.04902	57	0.849	23
2680	2708.7	64.2	107.25	3.20	0.04959	58	0.872	24
2660	2772.9	64.5	110.45	3.25	0.05017	59	0.896	25
2640	2837.4	64.7	113.70	3.30	0.05076	60	0.921	24
2620	2902.1	65.0	117.00	3.36	0.05136	61	0.945	25
2600	2967.1	32.6	120.36	1.70	0.05197	31	0.970	13
2590	2999.7	32.7	122.08	1.72	0.05228	32	0.983	12
2580	3032.4	32.9	123.80	1.73	0.05260	31	0.995	13
2570	3065.3	32.9	125.53	1.75	0.05291	32	1.008	13
2560	3098.2	33.0	127.28	1.76	0.05323	33	1.021	13
2550	3131.2	33.1	129.04	1.77	0.05356	33	1.034	13
2540	3164.3	33.2	130.81	1.78	0.05389	33	1.047	13
2530	3197.5	33.3	132.59	1.81	0.05422	34	1.060	13
2520	3230.8	33.3	134.40	1.83	0.05456	34	1.073	14
2510	3264.1	33.5	136.23	1.84	0.05490	34	1.087	13
2500	3297.6	33.6	138.07	1.86	0.05524	35	1.100	13
2490	3331.2	33.6	139.93	1.88	0.05559	35	1.113	14
2480	3364.8	33.8	141.81	1.89	0.05594	35	1.127	14
2470	3398.6	33.8	143.70	1.91	0.05629	36	1.141	13
2460	3432.4	34.0	145.61	1.93	0.05665	36	1.154	14
2450	3466.4	34.0	147.54	1.95	0.05701	37	1.168	14
2440	3500.4	34.2	149.49	1.97	0.05738	37	1.182	14
2430	3534.6	34.2	151.46	1.98	0.05775	38	1.196	14
2420	3568.8	34.3	153.44	2.00	0.05813	38	1.210	15
2410	3603.1	34.5	155.44	2.02	0.05851	38	1.225	14
2400	3637.6	34.5	157.46	2.04	0.05889	39	1.239	14
2390	3672.1	34.6	159.50	2.06	0.05928	39	1.253	15
2380	3706.7	34.8	161.56	2.08	0.05967	40	1.268	14
2370	3741.5	34.8	163.64	2.10	0.06007	40	1.282	15
2360	3776.3	35.0	165.74	2.12	0.06047	40	1.297	15
2350	3811.3	35.0	167.86	2.14	0.06087	41	1.312	15
2340	3846.3	35.2	170.00	2.16	0.06128	42	1.327	15
2330	3881.5	35.3	172.16	2.18	0.06170	42	1.342	15
2320	3916.8	35.3	174.34	2.21	0.06212	42	1.357	16
2310	3952.1	35.5	176.55	2.23	0.06254	43	1.373	15
2300	3987.6	35.6	178.78	2.25	0.06297	43	1.388	15
2290	4023.2	35.7	181.03	2.27	0.06340	44	1.403	16
2280	4058.9	35.8	183.30	2.30	0.06384	44	1.419	16
2270	4094.7	35.9	185.60	2.32	0.06428	45	1.435	15
2260	4130.6	36.0	187.92	2.34	0.06473	46	1.450	16
2250	4166.6	36.2	190.26	2.36	0.06519	46	1.466	16
2240	4202.8	36.2	192.62	2.39	0.06565	47	1.482	17
2230	4239.0	36.4	195.01	2.41	0.06612	47	1.499	16
2220	4275.4	36.4	197.42	2.44	0.06659	48	1.515	16
2210	4311.8	36.6	199.86	2.46	0.06707	48	1.531	17

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
2200	4345.4	36.7	202.32	2.50	0.06755	49	1.548	17
2190	4385.1	36.9	204.82	2.52	0.06804	50	1.565	16
2180	4422.0	36.9	207.34	2.54	0.06854	50	1.581	17
2170	4458.9	37.1	209.88	2.57	0.06904	51	1.598	17
2160	4496.0	37.2	212.45	2.60	0.06955	52	1.615	18
2150	4533.2	37.3	215.05	2.62	0.07007	52	1.633	17
2140	4570.5	37.4	217.67	2.65	0.07059	53	1.650	17
2130	4607.9	37.6	220.32	2.68	0.07112	53	1.667	18
2120	4645.5	37.6	223.00	2.70	0.07165	54	1.685	18
2110	4683.1	37.8	225.70	2.73	0.07219	55	1.703	18
2100	4720.9	38.0	228.43	2.77	0.07274	56	1.721	18
2090	4758.9	38.1	231.20	2.80	0.07330	56	1.739	18
2080	4797.0	38.2	234.00	2.83	0.07386	57	1.757	19
2070	4835.2	38.3	236.83	2.86	0.07443	58	1.776	18
2060	4873.5	38.4	239.69	2.90	0.07501	58	1.794	19
2050	4911.9	38.6	242.59	2.93	0.07559	60	1.813	19
2040	4950.5	38.7	245.52	2.96	0.07619	60	1.832	19
2030	4989.2	38.9	248.48	2.99	0.07679	61	1.851	19
2020	5028.1	38.9	251.47	3.03	0.07740	62	1.870	19
2010	5067.0	39.1	254.50	3.07	0.07802	62	1.889	20
2000	5106.1	39.2	257.57	3.10	0.07864	64	1.909	20
1990	5145.3	39.4	260.67	3.14	0.07928	64	1.929	19
1980	5184.7	39.6	263.81	3.17	0.07992	66	1.948	20
1970	5224.3	39.6	266.98	3.21	0.08058	66	1.968	21
1960	5263.9	39.8	270.19	3.25	0.08124	67	1.989	20
1950	5303.7	40.0	273.44	3.29	0.08191	68	2.009	21
1940	5343.7	40.1	276.73	3.32	0.08259	69	2.030	20
1930	5383.8	40.2	280.05	3.36	0.08328	70	2.050	21
1920	5424.0	40.4	283.41	3.40	0.08398	71	2.071	22
1910	5464.4	40.6	286.81	3.45	0.08469	71	2.093	21
1900	5505.0	40.7	290.26	3.49	0.08540	73	2.114	21
1890	5545.7	40.8	293.75	3.53	0.08613	74	2.135	22
1880	5586.5	41.0	297.28	3.58	0.08687	75	2.157	22
1870	5627.5	41.1	300.86	3.62	0.08762	76	2.179	22
1860	5668.6	41.3	304.48	3.67	0.08838	77	2.201	22
1850	5709.9	41.5	308.15	3.71	0.08915	78	2.223	23
1840	5751.4	41.6	311.86	3.76	0.08993	80	2.246	22
1830	5793.0	41.8	315.62	3.81	0.09073	80	2.268	23
1820	5834.8	41.9	319.43	3.85	0.09153	82	2.291	24
1810	5876.7	42.1	323.28	3.90	0.09235	83	2.315	23
1800	5918.8	42.3	327.18	3.96	0.09318	84	2.338	24
1790	5961.1	42.6	331.14	4.02	0.09402	86	2.362	24
1780	6003.7	42.8	335.16	4.08	0.09488	88	2.386	24
1770	6046.5	43.1	339.24	4.14	0.09576	89	2.410	24
1760	6089.6	43.3	343.38	4.21	0.09665	90	2.434	24

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
1750	6132.9	43.5	347.59	4.27	0.09755	92	2.458	25
1740	6176.4	43.8	351.86	4.33	0.09847	94	2.483	25
1730	6220.2	44.1	356.19	4.40	0.09941	95	2.508	26
1720	6264.3	44.3	360.59	4.47	0.10036	97	2.534	26
1710	6308.6	44.5	365.06	4.54	0.10133	99	2.560	26
1700	6353.1	44.8	369.60	4.61	0.10232	100	2.586	26
1690	6397.9	45.1	374.21	4.68	0.10332	102	2.612	27
1680	6443.0	45.4	378.89	4.76	0.10434	104	2.639	27
1670	6488.4	45.7	383.65	4.83	0.10538	106	2.666	27
1660	6534.1	45.9	388.48	4.91	0.10644	108	2.693	28
1650	6580.0	46.2	393.39	4.99	0.10752	110	2.721	28
1640	6626.2	46.5	398.38	5.08	0.10862	112	2.749	28
1630	6672.7	46.7	403.46	5.16	0.10974	113	2.777	29
1620	6719.4	47.1	408.62	5.24	0.11087	116	2.806	29
1610	6766.5	47.3	413.86	5.33	0.11203	119	2.835	30
1600	6813.8	23.8	419.19	2.70	0.11322	60	2.865	15
1595	6837.6	23.9	421.89	2.72	0.11382	60	2.880	15
1590	6861.5	23.9	424.61	2.75	0.11442	61	2.895	15
1585	6885.4	24.0	427.36	2.77	0.11503	62	2.910	16
1580	6909.4	24.1	430.13	2.79	0.11565	62	2.926	15
1575	6933.5	24.1	432.92	2.82	0.11627	63	2.941	15
1570	6957.6	24.3	435.74	2.84	0.11690	63	2.956	16
1565	6981.9	24.3	438.58	2.87	0.11753	64	2.972	15
1560	7006.2	24.4	441.45	2.89	0.11817	65	2.987	16
1555	7030.6	24.5	444.34	2.91	0.11882	65	3.003	16
1550	7055.1	24.6	447.25	2.94	0.11947	66	3.019	16
1545	7079.7	24.6	450.19	2.97	0.12013	67	3.035	16
1540	7104.3	24.7	453.16	2.99	0.12080	67	3.051	16
1535	7129.0	24.8	456.15	3.02	0.12147	68	3.067	16
1530	7153.8	24.9	459.17	3.05	0.12215	69	3.083	16
1525	7178.7	24.9	462.22	3.08	0.12284	69	3.099	16
1520	7203.6	25.0	465.30	3.10	0.12353	70	3.115	17
1515	7228.6	25.1	468.40	3.13	0.12423	71	3.132	16
1510	7253.7	25.2	471.53	3.16	0.12494	71	3.148	17
1505	7278.9	25.3	474.69	3.18	0.12565	72	3.165	17
1500	7304.2	25.5	477.87	3.21	0.12637	73	3.182	17
1495	7329.7	25.5	481.08	3.25	0.12710	73	3.199	17
1490	7355.2	25.6	484.33	3.27	0.12783	75	3.216	17
1485	7380.8	25.6	487.60	3.31	0.12858	75	3.233	18
1480	7406.4	25.7	490.91	3.33	0.12933	75	3.251	17
1475	7432.1	25.8	494.24	3.37	0.13008	77	3.268	18
1470	7457.9	25.9	497.61	3.40	0.13085	77	3.286	17
1465	7483.8	26.0	501.01	3.43	0.13162	78	3.303	18
1460	7509.8	26.1	504.44	3.46	0.13240	79	3.321	18
1455	7535.9	26.1	507.90	3.49	0.13319	80	3.339	18

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
1450	7562.0	26.3	511.39	3.53	0.13399	81	3.357	18
1445	7588.3	26.3	514.92	3.56	0.13480	81	3.375	18
1440	7614.6	26.4	518.48	3.59	0.13561	82	3.393	19
1435	7641.0	26.5	522.07	3.63	0.13643	83	3.412	18
1430	7667.5	26.7	525.70	3.67	0.13726	84	3.430	19
1425	7694.2	26.7	529.37	3.70	0.13810	85	3.449	18
1420	7720.9	26.8	533.07	3.74	0.13895	86	3.467	19
1415	7747.7	26.9	536.81	3.77	0.13981	87	3.486	20
1410	7774.6	26.9	540.58	3.81	0.14068	87	3.506	19
1405	7801.5	27.0	544.39	3.84	0.14155	89	3.525	19
1400	7828.5	27.2	548.23	3.88	0.14244	90	3.544	19
1395	7855.7	27.3	552.11	3.93	0.14334	90	3.563	20
1390	7883.0	27.4	556.04	3.96	0.14424	92	3.583	20
1385	7910.4	27.5	560.00	4.00	0.14516	92	3.603	20
1380	7937.9	27.6	564.00	4.05	0.14608	94	3.623	20
1375	7965.5	27.6	568.05	4.08	0.14702	94	3.643	20
1370	7993.1	27.9	572.13	4.15	0.14796	96	3.663	20
1365	8021.0	28.1	576.28	4.20	0.14892	97	3.683	21
1360	8049.1	28.4	580.48	4.27	0.14989	99	3.704	21
1355	8077.5	28.6	584.75	4.32	0.15088	101	3.725	21
1350	8106.1	28.8	589.07	4.38	0.15189	102	3.746	21
1345	8134.9	29.0	593.45	4.45	0.15291	103	3.767	22
1340	8163.9	29.2	597.90	4.51	0.15394	105	3.789	22
1335	8193.1	29.4	602.41	4.58	0.15499	107	3.811	22
1330	8222.5	29.6	606.99	4.64	0.15606	108	3.833	22
1325	8252.1	29.9	611.63	4.71	0.15714	110	3.855	23
1320	8282.0	30.1	616.34	4.78	0.15824	112	3.878	23
1315	8312.1	30.3	621.12	4.85	0.15936	113	3.901	23
1310	8342.4	30.6	625.97	4.92	0.16049	115	3.924	23
1305	8373.0	30.8	630.89	5.00	0.16164	116	3.947	24
1300	8403.8	12.4	635.89	2.02	0.16280	47	3.971	9
1298	8416.2	12.4	637.91	2.03	0.16327	48	3.980	10
1296	8428.6	12.5	639.94	2.04	0.16375	48	3.990	10
1294	8441.1	12.5	641.98	2.06	0.16423	48	4.000	9
1292	8453.6	12.5	644.04	2.07	0.16471	48	4.009	10
1290	8466.1	12.6	646.11	2.08	0.16519	49	4.019	10
1288	8478.7	12.6	648.19	2.09	0.16568	49	4.029	9
1286	8491.3	12.7	650.28	2.11	0.16617	49	4.038	10
1284	8504.0	12.7	652.39	2.12	0.16666	50	4.048	10
1282	8516.7	12.7	654.51	2.13	0.16716	50	4.058	10
1280	8529.4	12.8	656.64	2.14	0.16766	50	4.068	10
1278	8542.2	12.8	658.78	2.16	0.16816	51	4.078	10
1276	8555.0	12.9	660.94	2.17	0.16867	50	4.088	10
1274	8567.9	12.9	663.11	2.19	0.16917	51	4.098	10
1272	8580.8	12.9	665.30	2.20	0.16968	52	4.108	10

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
1270	8593.7	13.0	667.50	2.21	0.17020	52	4.118	10
1268	8606.7	13.0	669.71	2.23	0.17072	52	4.128	11
1266	8619.7	13.1	671.94	2.24	0.17124	53	4.139	10
1264	8632.8	13.1	674.18	2.25	0.17177	53	4.149	10
1262	8645.9	13.1	676.43	2.27	0.17230	53	4.159	11
1260	8659.0	13.2	678.70	2.28	0.17283	53	4.170	10
1258	8672.2	13.2	680.98	2.30	0.17336	54	4.180	11
1256	8685.4	13.2	683.28	2.31	0.17390	55	4.191	11
1254	8698.6	13.3	685.59	2.33	0.17445	54	4.202	10
1252	8711.9	13.4	687.92	2.34	0.17499	55	4.212	11
1250	8725.3	13.4	690.26	2.36	0.17554	56	4.223	11
1248	8738.7	13.4	692.62	2.37	0.17610	55	4.234	10
1246	8752.1	13.5	694.99	2.38	0.17665	56	4.244	11
1244	8765.6	13.5	697.37	2.40	0.17721	57	4.255	11
1242	8779.1	13.6	699.77	2.42	0.17778	56	4.266	11
1240	8792.7	13.6	702.19	2.43	0.17834	57	4.277	11
1238	8806.3	13.7	704.62	2.45	0.17891	57	4.288	11
1236	8820.0	13.7	707.07	2.46	0.17948	57	4.299	12
1234	8833.7	13.8	709.53	2.48	0.18005	58	4.311	11
1232	8847.5	13.8	712.01	2.49	0.18063	59	4.322	11
1230	8861.3	13.8	714.50	2.51	0.18122	59	4.333	11
1228	8875.1	13.9	717.01	2.54	0.18181	60	4.344	12
1226	8889.0	14.0	719.55	2.56	0.18241	60	4.356	11
1224	8903.0	14.1	722.11	2.59	0.18301	61	4.367	11
1222	8917.1	14.2	724.70	2.61	0.18362	61	4.378	12
1220	8931.3	14.3	727.31	2.64	0.18423	62	4.390	12
1218	8945.6	14.4	729.95	2.67	0.18485	62	4.402	11
1216	8960.0	14.5	732.62	2.69	0.18547	63	4.413	12
1214	8974.5	14.6	735.31	2.72	0.18610	64	4.425	12
1212	8989.1	14.7	738.03	2.74	0.18674	64	4.437	12
1210	9003.8	14.7	740.77	2.77	0.18738	65	4.449	12
1208	9018.5	14.9	743.54	2.80	0.18803	66	4.461	13
1206	9033.4	15.0	746.34	2.83	0.18869	66	4.474	12
1204	9048.4	15.0	749.17	2.86	0.18935	67	4.486	12
1202	9063.4	15.2	752.03	2.89	0.19002	68	4.498	13
1200	9078.6	15.3	754.92	2.92	0.19070	68	4.511	13
1198	9093.9	15.4	757.84	2.94	0.19138	69	4.524	13
1196	9109.3	15.4	760.78	2.98	0.19207	70	4.537	13
1194	9124.7	15.6	763.76	3.01	0.19277	71	4.550	13
1192	9140.3	15.7	766.77	3.04	0.19348	71	4.563	13
1190	9156.0	15.8	769.81	3.07	0.19419	72	4.576	13
1188	9171.8	15.9	772.88	3.11	0.19491	72	4.589	14
1186	9187.7	16.0	775.99	3.14	0.19563	73	4.603	13
1184	9203.7	16.1	779.13	3.17	0.19636	74	4.616	14
1182	9219.8	16.2	782.30	3.20	0.19710	75	4.630	14

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
1180	9236.0	16.3	785.50	3.24	0.19785	75	4.644	14
1178	9252.3	16.5	788.74	3.27	0.19860	76	4.658	14
1176	9268.8	16.5	792.01	3.31	0.19936	77	4.672	14
1174	9285.3	16.7	795.32	3.34	0.20013	78	4.686	14
1172	9302.0	16.8	798.66	3.38	0.20091	79	4.700	14
1170	9318.8	16.9	802.04	3.42	0.20170	80	4.714	15
1168	9335.7	17.0	805.46	3.45	0.20250	80	4.729	14
1166	9352.7	17.2	808.91	3.49	0.20330	82	4.743	15
1164	9369.9	17.2	812.40	3.53	0.20412	82	4.758	15
1162	9387.1	17.4	815.93	3.57	0.20494	83	4.773	15
1160	9404.5	17.5	819.50	3.60	0.20577	84	4.788	15
1158	9422.0	17.6	823.10	3.65	0.20661	84	4.803	16
1156	9439.6	17.7	826.75	3.68	0.20745	86	4.819	15
1154	9457.3	17.9	830.43	3.73	0.20831	87	4.834	15
1152	9475.2	18.0	834.16	3.77	0.20918	87	4.849	16
1150	9493.2	18.1	837.93	3.81	0.21005	88	4.865	16
1148	9511.3	18.2	841.74	3.86	0.21093	89	4.881	16
1146	9529.5	18.4	845.60	3.90	0.21182	90	4.897	16
1144	9547.9	18.5	849.50	3.94	0.21272	91	4.913	16
1142	9566.4	18.6	853.44	3.98	0.21363	92	4.929	16
1140	9585.0	18.7	857.42	4.03	0.21455	93	4.945	17
1138	9603.7	18.9	861.45	4.08	0.21548	94	4.962	16
1136	9622.6	19·0	865.53	4.13	0.21642	95	4.978	17
1134	9641.6	19.2	869.66	4.17	0.21737	96	4.995	17
1132	9660.8	19.3	873.83	4.22	0.21833	97	5.012	17
1130	9680.1	19.4	878.05	4.27	0.21930	98	5.029	17
1128	9699.5	19.6	882.32	4.32	0.22028	99	5.046	17
1126	9719.1	19.7	886.64	4.38	0.22127	100	5.063	18
1124	9738.8	19.8	891.02	4.42	0.22227	102	5.081	17
1122	9758.6	20.0	895.44	4.48	0.22329	102	5.098	18
1120	9778.6	20.1	899.92	4.53	0.22431	103	5.116	18
1118	9798.7	20.3	904.44	4.58	0.22534	105	5.134	18
1116	9819.0	20.4	909.02	4.63	0.22639	105	5.152	19
1114	9839.4	20.6	913.65	4.69	0.22744	107	5.171	18
1112	9860.0	20.7	918.34	4.74	0.22851	108	5.189	19
1110	9880.7	20.9	923.08	4.80	0.22959	109	5.208	19
1108	9901.6	21.0	927.88	4.86	0.23068	110	5.227	19
1106	9922.6	21.2	932.74	4.91	0.23178	112	5.246	19
1104	9943.8	21.3	937.65	4.98	0.23290	113	5.265	19
1102	9965.1	21.5	942.63	5.04	0.23403	114	5.284	20
1100	9986.6	10.8	947.67	2.54	0.23517	57	5.304	10
1099	9997.4	10.9	950.21	2.56	0.23574	58	5.314	10
1098	10008.3	10.9	952.77	2.58	0.23632	58	5.324	9
1097	10019.2	10.9	955.35	2.59	0.23690	59	5.333	10
1096	10030.1	10.9	957.94	2.60	0.23749	58	5.343	10

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
1095	10041.0	11.0	960.54	2.62	0.23807	59	5.353	10
1094	10052.0	11.0	963.16	2.64	0.23866	60	5.363	11
1093	10063.0	11.1	965.80	2.65	0.23926	60	5.374	10
1092	10074.1	11.1	968.45	2.67	0.23986	60	5.384	10
1091	10085.2	11.2	971.12	2.69	0.24046	60	5.394	10
1090	10096.4	11.2	973.81	2.70	0.24106	61	5.404	10
1089	10107.6	11.2	976.51	2.72	0.24167	61	5.414	11
1088	10118.8	11.3	979.23	2.74	0.24228	61	5.425	10
1087	10130.1	11.3	981.97	2.75	0.24289	62	5.435	10
1086	10141.4	11.3	984.72	2.77	0.24351	62	5.445	11
1085	10152.7	11.4	987.49	2.79	0.24413	62	5.456	10
1084	10164.1	11.5	990.28	2.81	0.24475	63	5.466	11
1083	10175.6	11.5	993.09	2.82	0.24538	63	5.477	11
1082	10187.1	11.5	995.91	2.84	0.24601	63	5.488	10
1081	10198.6	11.6	998.75	2.86	0.24664	64	5.498	11
1080	10210.2	11.6	1001.61	2.88	0.24728	64	5.509	11
1079	10221.8	11.7	1004.49	2.89	0.24792	65	5.520	11
1078	10233.5	11.7	1007.38	2.92	0.24857	65	5.531	10
1077	10245.2	11.8	1010.30	2.93	0.24922	65	5.541	11
1076	10257.0	11.8	1013.23	2.95	0.24987	65	5.552	11
1075	10268.8	11.8	1016.18	2.97	0.25052	66	5.563	11
1074	10280.6	11.9	1019.15	2.99	0.25118	67	5.574	12
1073	10292.5	11.9	1022.14	3.01	0.25185	66	5.586	11
1072	10304.4	12.0	1025.15	3.02	0.25251	67	5.597	11
1071	10316.4	12.0	1028.17	3.05	0.25318	68	5.608	11
1070	10328.4	12.1	1031.22	3.07	0.25386	68	5.619	11
1069	10340.5	12.1	1034.29	3.08	0.25454	68	5.630	12
1068	10352.6	12.1	1037.37	3.11	0.25522	68	5.642	11
1067	10364.7	12.2	1040.48	3.13	0.25590	69	5.653	11
1066	10376.9	12.3	1043.61	3.14	0.25659	69	5.664	12
1065	10389.2	12.3	1046.75	3.17	0.25728	70	5.676	11
1064	10401.5	12.3	1049.92	3.18	0.25798	70	5.687	12
1063	10413.8	12.4	1053.10	3.21	0.25868	71	5.699	11
1062	10426.2	12.4	1056.31	3.23	0.25939	71	5.710	12
1061	10438.6	12.5	1059.54	3.25	0.26010	71	5.722	12
1060	10451.1	12.5	1062.79	3.27	0.26081	72	5.734	12
1059	10463.6	12.6	1066.06	3.29	0.26153	72	5.746	12
1058	10476.2	12.6	1069.35	3.31	0.26225	73	5.758	12
1057	10488.8	12.7	1072.66	3.34	0.26298	73	5.770	12
1056	10501.5	12.7	1076.00	3.36	0.26371	73	5.782	12
1055	10514.2	12.8	1079.36	3.38	0.26444	74	5.794	12
1054	10527.0	12.8	1082.74	3.40	0.26518	74	5.806	12
1053	10539.8	12.8	1086.14	3.42	0.26592	75	5.818	13
1052	10552.6	12.9	1089.56	3.45	0.26667	75	5.830	12
1051	10565.5	13.0	1093.01	3.47	0.26742	76	5.843	12

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
1050	10578.5	13.0	1096.48	3.49	0.26818	76	5.855	12
1049	10591.5	13.0	1099.97	3.52	0.26894	76	5.867	13
1048	10604.5	13.1	1103.49	3.54	0.26970	77	5.880	12
1047	10617.6	13.2	1107.03	3.57	0.27047	77	5.892	13
1046	10630.8	13.2	1110.60	3.59	0.27124	78	5.905	12
1045	10644.0	13.2	1114.19	3.61	0.27202	78	5.911	13
1044	10657.2	13.3	1117.80	3.63	0.27280	78	5.930	12
1043	10670.5	13.4	1121.43	3.66	0.27358	79	5.942	13
1042	10683.9	13.4	1125.09	3.69	0.27437	80	5.955	13
1041	10697.3	13.5	1128.78	3.71	0.27517	80	5.968	13
1040	10710.8	13.5	1132.49	3.73	0.27597	80	5.981	13
1039	10724.3	13.6	1136.22	3.76	0.27677	81	5.994	13
1038	10737.9	13.6	1139.98	3.79	0.27758	82	6.007	13
1037	10751.5	13.7	1143.77	3.81	0.27840	82	6.020	13
1036	10765.2	13.8	1147.58	3.84	0.27922	82	6.033	14
1035	10779.0	13.8	1151.42	3.87	0.28004	83	6.047	13
1034	10792.8	13.8	1155.29	3.89	0.28087	83	6.060	13
1033	10806.6	13.9	1159.18	3.92	0.28170	84	6.073	14
1032	10820.5	13.9	1163.10	3.94	0.28254	84	6.087	13
1031	10834.4	14.0	1167.04	3.97	0.28338	85	6.100	14
1030	10848.4	14.0	1171.01	4.00	0.28423	85	6.114	14
1029	10862.4	14.1	1175.01	4.03	0.28508	86	6.128	13
1028	10876.5	14.2	1179.04	4.06	0.28594	86	6.141	14
1027	10890.7	14.2	1183.10	4.08	0.28680	87	6.155	14
1026	10904.9	14.2	1187.18	4.11	0.28767	87	6.169	14
1025	10919.1	14.3	1191.29	4.14	0.28854	88	6.183	14
1024	10933.4	14.4	1195.43	4.17	0.28942	88	6.197	14
1023	10947.8	14.4	1199.60	4.19	0.29030	89	6.211	15
1022	10962.2	14.5	1203.79	4.23	0.29119	89	6.226	14
1021	10976.7	14.6	1208.02	4.25	0.29208	90	6.240	14
1020	10991.3	14.6	1212.27	4.29	0.29298	90	6.254	14
1019	11005.9	14.7	1216.56	4.31	0.29388	91	6.268	15
1018	11020.6	14.7	1220.87	4.35	0.29479	92	6.283	14
1017	11035.3	14.8	1225.22	4.38	0.29571	92	6.297	14
1016	11050.1	14.8	1229.60	4.41	0.29663	93	6.311	15
1015	11064.9	14.9	1234.01	4.44	0.29756	93	6.326	15
1014	11079.8	15.0	1238.45	4.47	0.29849	94	6.341	14
1013	11094.8	15.0	1242.92	4.51	0.29943	94	6.355	15
1012	11109.8	15.1	1247.43	4.53	0.30037	95	6.370	15
1011	11124.9	15.1	1251.96	4.57	0.30132	95	6.385	15
1010	11140.0	15.2	1256.53	4.60	0.30227	96	6.400	15
1009	11155.2	15.2	1261.13	4.63	0.30323	96	6.415	15
1008	11170.4	15.3	1265.76	4.67	0.30419	97	6.430	15
1007	11185.7	15.4	1270.43	4.70	0.30516	98	6.445	16
1006	11201.1	15.4	1275.13	4.73	0.30614	98	6.461	15

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
1005	11216.5	15.5	1279.86	4.77	0.30712	99	6.476	15
1004	11232.0	15.6	1284.63	4.80	0.30811	99	6.491	16
1003	11247.6	16.6	1289.43	4.84	0.30910	100	6.507	15
1002	11263.2	15.7	1294.27	4.87	0.31010	101	6.522	16
1001	11278.9	15.8	1299.14	4.91	0.31111	101	6.538	16
1000	11294.7	15.8	1304.05	4.94	0.31212	102	6.554	16
999	11310.5	15.9	1308.99	4.98	0.31314	102	6.570	16
998	11326.4	16.0	1313.97	5.02	0.31416	103	6.586	16
997	11342.4	16.0	1318.99	5.05	0.31519	104	6.602	16
996	11358.4	16.1	1324.04	5.09	0.31623	104	6.618	16
995	11374.5	16.1	1329.13	5.12	0.31727	105	6.634	16
994	11390.6	16.2	1334.25	5.17	0.31832	105	6.650	17
993	11406.8	16.3	1339.42	5.20	0.31937	106	6.667	16
992	11423.1	16.3	1344.62	5.24	0.32043	107	6.683	16
991	11439.4	16.4	1349.86	5.28	0.32150	108	6.699	17
990	11455.8	16.5	1355.14	5.32	0.32258	108	6.716	17
989	11472.3	16.5	1360.46	5.36	0.32366	109	6.733	16
988	11488.8	16.6	1365.82	5.40	0.32475	109	6.749	17
987	11505.4	16.6	1371.22	5.44	0.32584	110	6.766	17
986	11522.0	16.7	1376.66	5.48	0.32694	111	6.783	17
985	11538.7	16.8	1382.14	5.52	0.32805	112	6.800	17
984	11555.5	16.9	1387.66	5.56	0.32917	112	6.817	17
983	11572.4	16.9	1393.22	5.61	0.33029	113	6.834	17
982	11589.3	17.0	1398.83	5.64	0.33142	114	6.851	18
981	11606.3	17.1	1404.47	5.69	0.33256	114	6.869	17
980	11623.4	17.2	1410.16	5.73	0.33370	115	6.886	17
979	11640.6	17.2	1415.89	5.77	0.33485	115	6.903	18
978	11657.8	17.3	1421.66	5.82	0.33600	117	6.921	18
977	11675.1	17.3	1427.48	5.86	0.33717	117	6.939	18
976	11692.4	17.4	1433.34	5.91	0.33834	118	6.957	17
975	11709.8	17.5	1439.25	5.95	0.33952	118	6.974	18
974	11727.3	17.6	1445.20	6.00	0.34070	119	6.992	18
973	11744.9	17.7	1451.20	6.04	0.34189	120	7.010	19
972	11762.6	17.7	1457.24	6.09	0.34309	121	7.029	18
971	11780.3	17.8	1463.33	6.13	0.34430	122	7.047	18
970	11798.1	17.9	1469.46	6.19	0.34552	122	7.065	19
969	11816.0	18.0	1475.65	6.24	0.34674	123	7.084	18
968	11834.0	18.0	1481.89	6.28	0.34797	124	7.102	19
967	11852.0	18.0	1488.17	6.31	0.34921	124	7.121	19
966	11870.0	18.1	1494.48	6.35	0.35045	125	7.140	18
965	11888.1	18.1	1500.83	6.38	0.35170	125	7.158	19
964	11906.2	18.1	1507.21	6.42	0.35295	126	7.177	19
963	11924.3	18.2	1513.63	6.45	0.35421	126	7.196	19
962	11942.5	18.2	1520.08	6.49	0.35547	127	7.215	19
961	11960.7	18.3	1526.57	6.53	0.35674	127	7.234	19

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
960	11979.0	18.3	1533.10	6.56	0.35801	128	7.253	19
959	11997.3	18.3	1539.66	6.61	0.35929	128	7.272	19
958	12015.6	18.4	1546.27	6.64	0.36057	129	7.291	19
957	12034.0	18.4	1552.91	6.67	0.36186	129	7.310	19
956	12052.4	18.4	1559.58	6.72	0.36315	130	7.329	20
955	12070.8	18.5	1566.30	6.75	0.36445	131	7.349	19
954	12089.3	18.6	1573.05	6.79	0.36576	131	7.368	19
953	12107.9	18.5	1579.84	6.83	0.36707	132	7.387	20
952	12126.4	18.6	1586.67	6.87	0.36839	132	7.407	19
951	12145.0	18.7	1593.54	6.90	0.36971	133	7.426	20
950	12163.7	18.7	1600.44	6.94	0.37104	133	7.446	20
949	12182.4	18.7	1607.38	6.99	0.37237	134	7.466	19
948	12201.1	18.8	1614.37	7.03	0.37371	134	7.485	20
947	12219.9	18.8	1621.40	7.06	0.37505	135	7.505	20
946	12238.7	18.8	1628.46	7.11	0.37640	136	7.525	20
945	12257.5	18.9	1635.57	7.15	0.37776	136	7.545	20
944	12276.4	19.0	1642.72	7.19	0.37912	137	7.565	20
943	12295.4	18.9	1649.91	7.23	0.38049	137	7.585	21
942	12314.3	19.0	1657.14	7.27	0.38186	138	7.606	20
941	12333.3	19.1	1664.41	7.31	0.38324	139	7.626	20
940	12352.4	19.1	1671.72	7.36	0.38463	139	7.646	20
939	12371.5	19.1	1679.08	7.40	0.38602	140	7.666	21
938	12390.6	19.2	1686.48	7.44	0.38742	140	7.687	20
937	12409.8	19.2	1693.92	7.48	0.38882	141	7.707	21
936	12429.0	19.2	1701.40	7.53	0.39023	142	7.728	20
935	12448.2	19.3	1708.93	7.57	0.39165	142	7.748	21
934	12467.5	19.3	1716.50	7.62	0.39307	143	7.769	20
933	12486.8	19.4	1724.12	7.66	0.39450	143	7.789	21
932	12506.2	19.4	1731.78	7.70	0.39593	144	7.810	21
931	12525.6	19.5	1739.48	7.74	0.39737	145	7.831	21
930	12545.1	19.5	1747.22	7.79	0.39882	145	7.852	21
929	12564.6	19.6	1755.01	7.84	0.40027	146	7.873	21
928	12584.2	19.6	1762.85	7.88	0.40173	146	7.894	21
927	12603.8	19.6	1770.73	7.92	0.40319	147	7.915	21
926	12623.4	19.7	1778.65	7.98	0.40466	148	7.936	22
925	12643.1	19.7	1786.63	8.02	0.40614	148	7.958	21
924	12662.8	19.8	1794.65	8.07	0.40762	149	7.979	21
923	12682.6	19.8	1802.72	8.11	0.40911	150	8.000	22
922	12702.4	19.8	1810.83	8.16	0.41061	150	8.022	21
921	12722.2	19.9	1818.99	8.21	0.41211	151	8.043	22
920	12742.1	19.9	1827.20	8.25	0.41362	152	8.065	22
919	12762.0	20.0	1835.45	8.31	0.41514	152	8.087	21
918	12782.0	20.0	1843.76	8.36	0.41666	153	8.108	22
917	12802.0	20.0	1852.12	8.40	0.41819	153	8.130	22
916	12822.0	20.1	1860.52	8.45	0.41972	154	8.152	22

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
915	12842.1	20.2	1868.97	8.51	0.42126	155	8.174	22
914	12862.3	20.2	1877.48	8.55	0.42281	156	8.196	22
913	12882.5	20.2	1886.03	8.60	0.42437	156	8.218	23
912	12902.7	20.3	1894.63	8.66	0.42593	157	8.241	22
911	12923.0	20.3	1903.29	8.70	0.42750	158	8.263	22
910	12943.3	20.4	1911.99	8.76	0.42908	158	8.285	22
909	12963.7	20.4	1920.75	8.81	0.43066	159	8.307	23
908	12984.1	20.4	1929.56	8.86	0.43225	160	8.330	22
907	13004.5	20.5	1938.42	8.91	0.43385	161	8.352	23
906	13025.0	20.6	1947.33	8.97	0.43546	161	8.375	22
905	13045.6	20.6	1956.30	9.02	0.43707	162	8.397	23
904	13066.2	20.6	1965.32	9.07	0.43869	163	8.420	23
903	13086.8	20.7	1974.39	9.12	0.44032	163	8.443	23
902	13107.5	20.7	1983.51	9.18	0.44195	164	8.466	23
901	13128.2	20.8	1992.69	9.23	0.44359	165	8.489	23
900	13149.0	20.8	2001.92	9.28	0.44524	165	8.512	23
899	13169.8	20.9	2011.20	9.34	0.44689	167	8.535	23
898	13190.7	20.9	2020.54	9.40	0.44856	167	8.558	24
897	13211.6	21.0	2029.94	9.46	0.45023	167	8.582	23
896	13232.6	21.0	2039.40	9.51	0.45190	169	8.605	23
895	13253.6	21.1	2048.91	9.57	0.45359	169	8.628	24
894	13274.7	21.1	2058.48	9.63	0.45527	170	8.652	24
893	13295.8	21.1	2068.11	9.68	0.45698	171	8.676	23
892	13316.9	21.2	2077.79	9.74	0.45869	172	8.699	24
891	13338.1	21.3	2087.53	9.80	0.46041	172	8.723	24
890	13359.4	21.3	2097.33	9.86	0.46213	173	8.747	24
889	13380.7	21.3	2107.19	9.92	0.46386	174	8.771	24
888	13402.0	21.4	2117.11	9.98	0.46560	174	8.795	24
887	13423.4	21.5	2127.09	10.04	0.46734	176	8.819	24
886	13444.9	21.5	2137.13	10.10	0.46910	176	8.843	25
885	13466.4	21.5	2147.23	10.17	0.47086	177	8.868	24
884	13487.9	21.6	2157.40	10.22	0.47263	177	8.892	24
883	13509.5	21.6	2167.62	10.28	0.47440	179	8.916	25
882	13531.1	21.7	2177.90	10.34	0.47619	180	8.941	24
881	13552.8	21.7	2188.24	10.41	0.47799	180	8.965	25
880	13574.5	21.8	2198.65	10.47	0.47979	181	8.990	25
879	13596.3	21.8	2209.12	10.53	0.48160	182	9.015	25
878	13618.1	21.9	2219.65	10.60	0.48342	183	9.040	25
877	13640.0	21.9	2230.25	10.66	0.48525	183	9.065	25
876	13661.9	22.0	2240.91	10.73	0.48708	185	9.090	25
875	13683.9	22.0	2251.64	10.79	0.48893	185	9.115	25
874	13705.9	22.1	2262.43	10.86	0.49078	186	9.140	26
873	13728.0	22.1	2273.29	10.92	0.49264	187	9.166	25
872	13750.1	22.2	2284.21	10.99	0.49451	188	9.191	25
871	13772.3	22.2	2295.20	11.06	0.49639	189	9.216	26

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
870	13794.5	22.3	2306.26	11.12	0.49828	190	9.242	26
869	13816.8	22.3	2317.38	11.19	0.50018	190	9.268	25
868	13839.1	22.4	2328.57	11.26	0.50208	191	9.293	26
867	13861.5	22.4	2339.83	11.33	0.50399	192	9.319	26
866	13883.9	22.5	2351.16	11.41	0.50591	193	9.345	26
865	13906.4	22.6	2362.57	11.47	0.50784	195	9.371	26
864	13929.0	22.6	2374.04	11.54	0.50979	195	9.397	26
863	13951.6	22.6	2385.58	11.61	0.51174	195	9.423	26
862	13974.2	22.7	2397.19	11.69	0.51369	197	9.449	26
861	13996.9	22.8	2408.88	11.75	0.51566	198	9.475	27
860	14019.7	22.8	2420.63	11.83	0.51764	198	9.502	27
859	14042.5	22.9	2432.46	11.90	0.51962	200	9.529	26
858	14065.4	22.9	2444.36	11.98	0.52162	200	9.555	27
857	14088.3	23.0	2456.34	12.05	0.52362	202	9.582	27
856	14111.3	23.0	2468.39	12.13	0.52564	202	9.609	27
855	14134.3	23.1	2480.52	12.20	0.52766	203	9.636	27
854	14157.4	23.1	2492.72	12.28	0.52969	204	9.663	27
853	14180.5	23.2	2505.00	12.35	0.53173	206	9.690	27
852	14203.7	23.2	2517.35	12.43	0.53379	206	9.717	28
851	14226.9	23.3	2529.78	12.50	0.53585	207	9.745	27
850	14250.2	23.3	2542.28	12.58	0.53792	208	9.772	27
849	14273.5	23.4	2554.86	12.66	0.54000	209	9.799	28
848	14296.9	23.5	2567.52	12.74	0.54209	210	9.827	28
847	14320.4	23.5	2580.26	12.83	0.54419	211	9.855	27
846	14343.9	23.6	2593.09	12.90	0.54630	212	9.882	28
845	14367.5	23.6	2605.99	12.98	0.54842	213	9.910	28
844	14391.1	23.7	2618.97	13.07	0.55055	214	9.938	28
843	14414.8	23.7	2632.04	13.14	0.55269	216	9.966	28
842	14438.5	23.8	2645.18	13.22	0.55485	216	9.994	29
841	14462.3	23.9	2658.40	13.31	0.55701	217	10.023	28
840	14486.2	23.9	2671.71	13.39	0.55918	218	10.051	28
839	14510.1	23.9	2685.10	13.48	0.56136	219	10.079	29
838	14534.0	24.0	2698.58	13.56	0.56355	221	10.108	28
837	14558.0	24.1	2712.14	13.65	0.56576	221	10.136	29
836	14582.1	24.1	2725.79	13.74	0.56797	222	10.165	29
835	14606.2	24.2	2739.53	13.82	0.57019	224	10.194	29
834	14630.4	24.3	2753.35	13.91	0.57243	225	10.223	29
833	14654.7	24.3	2767.26	14.00	0.57468	225	10.252	29
832	14679.0	24.4	2781.26	14.09	0.57693	227	10.281	30
831	14703.4	24.4	2795.35	14.17	0.57920	228	10.311	29
830	14727.8	24.5	2809.52	14.26	0.58148	229	10.340	29
829	14752.3	24.5	2823.78	14.36	0.58377	229	10.369	30
828	14776.8	24.6	2838.14	14.45	0.58606	231	10.399	30
827	14801.4	24.7	2852.59	14.54	0.58837	233	10.429	30
826	14826.1	24.7	2867.13	14.64	0.59070	233	10.459	30

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
825	14850.8	24.8	2881.77	14.73	0.59303	234	10.489	30
824	14875.6	24.9	2896.50	14.82	0.59537	236	10.519	30
823	14900.5	24.9	2911.32	14.92	0.59773	237	10.549	30
822	14925.4	25.0	2926.24	15.01	0.60010	238	10.579	30
821	14950.4	25.0	2941.25	15.10	0.60248	239	10.609	31
820	14975.4	25.1	2956.35	15.20	0.60487	240	10.640	31
819	15000.5	25.1	2971.55	15.30	0.60727	241	10.671	30
818	15025.6	25.2	2986.85	15.41	0.60968	243	10.701	31
817	15050.8	25.3	3002.26	15.50	0.61211	244	10.732	31
816	15076.1	25.4	3017.76	15.60	0.61455	245	10.763	32
815	15101.5	25.4	3033.36	15.71	0.61700	246	10.795	31
814	15126.9	25.5	3049.07	15.80	0.61946	248	10.826	31
813	15152.4	25.5	3064.87	15.91	0.62194	249	10.857	32
812	15177.9	25.6	3080.78	16.00	0.62443	250	10.889	31
811	15203.5	25.6	3096.78	16.11	0.62693	251	10.920	32
810	15229.1	25.7	3112.89	16.21	0.62944	252	10.952	32
809	15254.8	25.8	3129.10	16.32	0.63196	254	10.984	32
808	15280.6	25.8	3145.42	16.43	0.63450	255	11.016	32
807	15306.4	25.9	3161.85	16.54	0.63705	256	11.048	32
806	15332.3	26.0	3178.39	16.65	0.63961	257	11.080	32
805	15358.3	26.0	3195.04	16.75	0.64218	259	11.112	32
804	15384.3	26.1	3211.79	16.86	0.64477	260	11.144	33
803	15410.4	26.2	3228.65	16.97	0.64737	261	11.177	32
802	15436.6	26.2	3245.62	17.08	0.64998	263	11.209	33
801	15462.8	26.3	3262.70	17.19	0.65261	264	11.242	33
800	15489.1	26.4	3279.89	17.30	0.65525	265	11.275	33
799	15515.5	26.4	3297.19	17.42	0.65790	266	11.308	33
798	15541.9	26.5	3314.61	17.53	0.66056	268	11.341	33
797	15568.4	26.5	3332.14	17.65	0.66324	269	11.374	33
796	15594.9	26.6	3349.79	17.77	0.66593	271	11.407	34
795	15621.5	26.7	3367.56	17.89	0.66864	272	11.441	33
794	15648.2	26.8	3385.45	18.00	0.67136	273	11.474	34
793	15675.0	26.8	3403.45	18.12	0.67409	275	11.508	34
792	15701.8	26 9	3421.57	18.24	0.67684	276	11.542	34
791	15728.7	26.9	3439.81	18.36	0.67960	278	11.576	34
790	15755.6	27.1	3458.17	18.52	0.68238	280	11.610	34
789	15782.7	27.1	3476.69	18.62	0.68518	280	11.644	35
788	15809.8	27.1	3495.31	18.71	0.68798	282	11.679	34
787	15836.9	27.2	3514.02	18.83	0.69080	283	11.713	35
786	15864.1	27.2	3532.85	18.92	0.69363	284	11.748	34
785	15891.3	27.3	3551.77	19.02	0.69647	285	11.782	35
784	15918.6	27.3	3570.79	19.13	0.69932	286	11.817	35
783	15945.9	27.3	3589.92	19.23	0.70218	287	11.852	35
782	15973.2	27.4	3609.15	19.33	0.70505	288	11.887	35
781	16000.6	27.4	3628.48	19.44	0.70793	289	11.922	35

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
780	16028.0	27.4	3647.92	19.54	0.71082	290	11.957	35
779	16055.4	27.5	3667.46	19.65	0.71372	292	11.992	35
778	16082.9	27.5	3687.11	19.75	0.71664	292	12.027	36
777	16110.4	27.5	3706.86	19.86	0.71956	294	12.063	35
776	16137.9	27.6	3726.72	19.97	0.72250	295	12.098	35
775	16165.5	27.6	3746.69	20.07	0.72545	296	12.133	36
774	16193.1	27.6	3766.76	20.18	0.72841	297	12.169	36
773	16220.7	27.7	3786.94	20.29	0.73138	299	12.205	36
772	16248.4	27.7	3807.23	20.39	0.73437	299	12.241	36
771	16276.1	27.8	3827.62	20.51	0.73736	301	12.277	36
770	16303.9	27.8	3848.13	20.62	0.74037	302	12.313	36
769	16331.7	27.8	3868.75	20.72	0.74339	303	12.349	37
768	16359.5	27.9	3889.47	20.84	0.74642	304	12.386	36
767	16387.4	27.9	3910.31	20.95	0.74946	306	12.422	36
766	16415.3	27.9	3931.26	21.07	0.75252	307	12.458	37
765	16443.2	28.0	3952.33	21.17	0.75559	308	12.495	37
764	16471.2	28.0	3973.50	21.29	0.75867	309	12.532	36
763	16499.2	28.1	3994.79	21.41	0.76176	310	12.568	37
762	16527.3	28.1	4016.20	21.52	0.76486	311	12.605	37
761	16555.4	28.1	4037.72	21.64	0.76797	313	12.642	37
760	16583.5	28.1	4059.36	21.76	0.77110	314	12.679	37
759	16611.6	28.2	4081.12	21.88	0.77424	315	12.716	37
758	16639.8	28.2	4103.00	22.00	0.77739	316	12.753	38
757	16668.0	28.3	4125.00	22.11	0.78055	318	12.791	37
756	16696.3	28.3	4147.11	22.23	0.78373	319	12.828	37
755	16724.6	28.3	4169.34	22.35	0.78692	320	12.865	38
754	16752.9	28.4	4191.69	22.47	0.79012	322	12.903	38
753	16781.3	28.4	4214.16	22.59	0.79334	322	12.941	37
752	16809.7	28.5	4236.75	22.70	0.79656	324	12.978	38
751	16838.2	28.5	4259.45	22.83	0.79980	326	13.016	38
750	16866.7	28.5	4282.28	22.96	0.80306	327	13.054	38
749	16895.2	28.6	4305.24	23.09	0.80633	328	13.092	38
748	16923.8	28.6	4328.33	23.21	0.80961	329	13.130	38
747	16952.4	28.7	4351.54	23.33	0.81290	331	13.168	39
746	16981.1	28.7	4374.87	23.46	0.81621	332	13.207	38
745	17009.8	28.7	4398.33	23.59	0.81953	333	13.245	39
744	17038.5	28.8	4421.92	23.72	0.82286	335	13.284	38
743	17067.3	28.8	4445.64	23.84	0.82621	336	13.322	39
742	17096.1	28.8	4469.48	23.98	0.82957	337	13.361	39
741	17124.9	28.9	4493.46	24.10	0.83294	339	13.400	39
740	17153.8	28.9	4517.56	24.23	0.83633	340	13.439	39
739	17182.7	29.0	4541.79	24.37	0.83973	341	13.478	39
738	17211.7	29.0	4566.16	24.50	0.84314	343	13.517	40
737	17240.7	29.0	4590.66	24.63	0.84657	344	13.557	39
736	17269.7	29.1	4615.29	24.76	0.85001	346	13.596	39

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
735	17298.8	29.1	4640.05	24.90	0.85347	347	13.635	40
734	17327.9	29.1	4664.95	25.04	0.85694	349	13.675	40
733	17357.0	29.2	4689.99	25.17	0.86043	350	13.715	40
732	17386.2	29.2	4715.16	25.30	0.86393	351	13.755	40
731	17415.4	29.3	4740.46	25.45	0.86744	353	13.795	40
730	17444.7	29.3	4765.91	25.59	0.87097	354	13.835	40
729	17474.0	29.3	4791.50	25.72	0.87451	356	13.875	41
728	17503.3	29.4	4817.22	25.87	0.87807	358	13.916	40
727	17532.7	29.5	4843.09	26.00	0.88165	359	13.956	40
726	17562.2	29.5	4869.09	26.15	0.88524	360	13.996	41
725	17591.7	29.5	4895.24	26.29	0.88884	361	14.037	41
724	17621.2	29.5	4921.53	26.43	0.89245	363	14.078	41
723	17650.7	29.6	4947.96	26.57	0.89608	365	14.119	41
722	17680.3	29.7	4974.53	26.72	0.89973	366	14.160	41
721	17710.0	29.7	5001.25	26.87	0 90339	368	14.201	41
720	17739.7	29.8	5028.12	27.01	0.90707	369	14.242	41
719	17769.5	29.8	5055.13	27.16	0.91076	371	14.283	42
718	17799.3	29.8	5082.29	27.31	0.91447	372	14.325	41
717	17829.1	29.9	5109.60	27.46	0.91819	374	14.366	42
716	17859.0	29.9	5137.06	27.61	0.92193	376	14.408	41
715	17888.9	30.0	5164.67	27.76	0.92569	377	14.449	42
714	17918.9	30.0	5192.43	27.92	0.92946	379	14.491	42
713	17948.9	30.0	5220.35	28.06	0.93325	380	14.533	43
712	17978.9	30.1	5248.41	28.22	0.93705	382	14.576	42
711	18009.0	30.1	5276.63	28.38	0.94087	383	14.618	42
710	18039.1	30.1	5305.01	28.53	0.94470	385	14.660	42
709	18069.2	30.1	5333.54	28.69	0.94855	387	14.702	43
708	18099.3	30.2	5362.23	28.85	0.95242	388	14.745	43
707	18129.5	30.3	5391.08	29.01	0.95630	390	14.788	42
706	18159.8	30.3	5420.09	29.16	0.96020	392	14.830	43
705	18190.1	30.3	5449.25	29.33	0.96412	394	14.873	43
704	18220.4	30.4	5478.58	29.49	0.96806	395	14.916	43
703	18250.8	30.4	5508.07	29.65	0.97201	397	14.959	44
702	18281.2	30.5	5537.72	29.81	0.97598	398	15.003	43
701	18311.7	30.5	5567.53	29.98	0.97996	400	15.046	44
700	18342.2	30.6	5597.51	30.15	0.98396	402	15.090	44
699	18372.8	30.6	5627.66	30.31	0.98798	404	15.134	43
698	18403.4	30.6	5657.97	30.48	0.99202	405	15.177	44
697	18434.0	30.7	5688.45	30.64	0.99607	407	15.221	45
696	18464.7	30.8	5719.09	30.82	1.00014	409	15.266	44
695	18495.5	30.8	5749.91	30.99	1.00423	411	15.310	44
694	18526.3	30.8	5780.90	31.15	1.00834	412	15.354	45
693	18557.1	30.9	5812.05	31.33	1.01246	414	15.399	44
692	18588.0	30.9	5843.38	31.50	1.01660	416	15.443	45
691	18618.9	31.0	5874.88	31.68	1.02076	418	15.488	45

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
690	18649.9	31.0	5906.56	31.85	1.02494	420	15.533	45
689	18680.9	31.0	5938.41	32.03	1.02914	421	15.578	45
688	18711.9	31.1	5970.44	32.21	1.03335	423	15.623	46
687	18743.0	31.1	6002.65	32.39	1.03758	425	15.669	45
686	18774.1	31.2	6035.04	32.57	1.04183	427	15.714	45
685	18805.3	31.3	6067.61	32.74	1.04610	429	15.759	46
684	18836.6	31.3	6100.35	32.93	1.05039	431	15.805	46
683	18867.9	31.3	6133.28	33.12	1.05470	433	15.851	46
682	18899.2	31.4	6166.40	33.30	1.05903	434	15.897	46
681	18930.6	31.4	6199.70	33.48	1.06337	437	15.943	46
680	18962.0	31.5	6233.18	33.67	1.06774	438	15.989	46
679	18993.5	31.5	6266.85	33.86	1.07212	441	16.035	47
678	19025.0	31.6	6300.71	34.05	1.07653	442	16.082	46
677	19056.6	31.6	6334.76	34.25	1.08095	445	16.128	47
676	19088.2	31.7	6369.01	34.43	1.08540	446	16.175	46
675	19119.9	31.7	6403.44	34.62	1.08986	448	16.221	47
674	19151.6	31.7	6438.06	34.82	1.09434	450	16.268	48
673	19183.3	31.8	6472.88	35.01	1.09884	453	16.316	47
672	19215.1	31.9	6507.89	35.21	1.10337	454	16.363	47
671	19247.0	31.9	6543.10	35.41	1.10791	456	16.410	48
670	19278.9	31.9	6578.51	35.61	1.11247	458	16.458	48
669	19310.8	32.0	6614.12	35.81	1.11705	460	16.506	48
668	19342.8	32.0	6649.93	36.01	1.12165	462	16.554	48
667	19374.8	32.1	6685.94	36.21	1.12627	465	16.602	48
666	19406.9	32.1	6722.15	36.42	1.13092	467	16.650	48
665	19439.0	32.2	6758.57	36.62	1.13559	468	16.698	49
664	19471.2	32.2	6795.19	36.83	1.14027	471	16.747	48
663	19503.4	32.3	6832.02	37.03	1.14498	473	16.795	49
662	19535.7	32.3	6869.05	37.24	1.14971	475	16.844	49
661	19568.0	32.4	6906.29	37.46	1.15446	478	16.893	49
660	19600.4	32.4	6943.75	37.67	1.15924	480	16.942	49
659	19632.8	32.5	6981.42	37.88	1.16404	482	16·991	49
658	19665.3	32.5	7019.30	38.10	1.16886	484	17.040	50
657	19697.8	32.6	7057.40	38.31	1.17370	486	17.090	49
656	19730.4	32.6	7095.71	38.53	1.17856	488	17.139	50
655	19763.0	32.7	7134.24	38.75	1.18344	491	17.189	50
654	19795.7	32.7	7172.99	38.97	1.18835	493	17.239	50
653	19828.4	32.8	7211.96	39.18	1.19328	495	17.289	50
652	19861.2	32.8	7251.14	39.41	1.19823	497	17.339	50
651	19894.0	32.9	7290.55	39.64	1.20320	500	17.389	51
650	19926.9	32.9	7330.19	39.86	1.20820	502	17.440	51
649	19959.8	33.0	7370.05	40.09	1.21322	504	17.491	51
648	19992.8	33.0	7410.14	40 32	1.21826	507	17.542	51
647	20025.8	33.1	7450.46	40.56	1.22333	509	17.593	51
646	20058.9	33.1	7491.02	40.78	1.22842	511	17.644	52

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
645	20092.0	33.2	7531.80	41.01	1.23353	514	17.696	51
644	20125.2	33.3	7572.81	41.25	1.23867	516	17.747	52
643	20158.5	33.3	7614.06	41.49	1.24383	518	17.799	52
642	20191.8	33.3	7655.55	41.72	1.24901	521	17.851	52
641	20225.1	33.4	7697.27	41.96	1.25422	524	17.903	52
640	20258.5	33.4	7739.23	42.21	1.25946	526	17.955	52
639	20291.9	33.5	7781.44	42.44	1.26472	528	18.007	53
638	20325.4	33.6	7823.88	42.69	1.27000	531	18.060	52
637	20359.0	33.6	7866.57	42.94	1.27531	534	18.112	53
636	20392.6	33.6	7909.51	43.19	1.28065	536	18.165	53
635	20426.2	33.7	7952.70	43.43	1.28601	538	18.218	53
634	20459.5	33.8	7996.13	43.68	1.29139	541	18.271	53
633	20493.7	33.8	8039.81	43.94	1.29680	544	18.324	54
632	20527.5	33.9	8083.75	44.18	1.30224	546	18.378	53
631	20561.4	33.9	8127.93	44.45	1.30770	549	18.431	54
630	20595.3	34.0	8172.38	44.70	1.31319	551	18.485	54
629	20629.3	34.0	8217.08	44.96	1.31870	554	18.539	54
628	20663.3	34.1	8262.04	45.22	1.32424	557	18.593	54
627	20697.4	34.1	8307.26	45.49	1.32981	560	18.647	54
626	20731.5	34.2	8352.75	45.75	1.33541	562	18.701	55
625	20765.7	34.3	8398.50	46.02	1.34103	564	18.756	55
624	20800.0	34.3	8444.52	46.28	1.34667	568	18.811	55
623	20834.3	34.3	8490.80	46.56	1.35235	570	18.866	55
622	20868.6	34.4	8537.36	46.83	1.35805	573	18.921	55
621	20903.0	34.5	8584.19	47.10	1.36378	576	18.976	56
620	20937.5	34.5	8631.29	47.38	1.36954	579	19.032	56
619	20972.0	34.6	8678.67	47.66	1.37533	581	19.088	56
618	21006.6	34.6	8726.33	47.93	1.38114	584	19.144	56
617	21041.2	34.7	8774.26	48.22	1.38698	587	19.200	56
616	21075.9	34.7	8822.48	48.49	1.39285	590	19.256	57
615	21110.6	34.8	8870.97	48.78	1.39875	593	19.313	56
614	21145.4	34.9	8919.75	49.07	1.40468	596	19.369	57
613	21180.3	34.9	8968.82	49.36	1.41064	598	19.426	57
612	21215.2	35.0	9018.18	49.64	1.41662	602	19.483	58
611	21250.2	35.0	9067.82	49.94	1.42264	604	19.541	57
610	21285.2	35.1	9117.76	50.23	1.42868	607	19.598	57
609	21320.3	35.1	9167.99	50.53	1.43475	611	19.655	58
608	21355.4	35.2	9218.52	50.82	1.44086	613	19.713	58
607	21390.6	35.3	9269.34	51.13	1.44699	617	19.771	58
606	21425.9	35.3	9320.47	51.43	1.45316	619	19.829	58
605	21461.2	35.4	9371.90	51.74	1.45935	623	19.887	59
604	21496.6	35.4	9423.64	52.04	1.46558	626	19.946	59
603	21532.0	35.5	9475.68	52.36	1.47184	629	20.005	59
602	21567.5	35.6	9528.04	52.66	1.47813	632	20.064	59
601	21603.1	35.6	9580.70	52.98	1.48445	635	20.123	59

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
600	21638.7	35.7	9633.68	53.30	1.49080	638	20.182	59
599	21674.4	35.7	9686.98	53.62	1.49718	642	20.241	60
598	21710.1	35.8	9740.60	53.94	1.50360	645	20.301	60
597	21745.9	35.8	9794.54	54.25	1.51005	648	20.361	60
596	21781.7	35.9	9848.79	54.58	1.51653	652	20.421	60
595	21817.6	36.0	9903.37	54.91	1.52305	654	20.481	61
594	21853.6	36.0	9958.28	55.23	1.52959	658	20.542	61
593	21889.6	36.1	10013.50	55.6	1.53617	661	20.603	61
592	21925.7	36.2	10069.10	55.9	1.54278	665	20.664	61
591	21961.9	36.2	10125.00	56.2	1.54943	668	20.725	61
590	21998.1	36.3	10181.20	56.6	1.55611	671	20.786	61
589	22034.4	36.3	10237.80	56.9	1.56282	675	20.847	62
588	22070.7	36.4	10294.70	57.2	1.56957	679	20.909	62
587	22107.1	36.4	10351.90	57.6	1.57636	681	20.971	62
586	22143.5	36.5	10409.50	58.0	1.58317	685	21.033	63
585	22180.0	36.6	10467.50	58.3	1.59002	689	21.096	62
584	22216.6	36.7	10525.80	58.6	1.59691	693	21.158	63
583	22253.3	36.7	10584.40	59.0	1.60384	696	21.221	63
582	22290.0	36.8	10643.40	59.4	1.61080	699	21.284	63
581	22326.8	36.8	10702.80	59.7	1.61779	703	21.347	64
580	22363.6	36.9	10762.50	60.1	1.62482	707	21.411	64
579	22400.5	37.0	10822.60	60.4	1.63189	710	21.475	64
578	22437.5	37.0	10883.00	60.9	1.63899	714	21.539	64
577	22474.5	37.1	10943.90	61.2	1.64613	718	21.603	64
576	22511.6	37.1	11005.10	61.5	1.65331	721	21.667	65
575	22548.7	37.2	11066.60	62.0	1.66052	726	21.732	64
574	22585.9	37.3	11128.60	62.3	1.66778	729	21.796	65
573	22623.2	37.4	11190.90	62.7	1.67507	733	21.861	66
572	22660.6	37.4	11253.60	63.1	1.68240	737	21.927	65
571	22698.0	37.5	11316.70	63.5	1.68977	741	21.992	66
570	22735.5	37.6	11380.20	63.9	1.69718	745	22.058	66
569	22773.1	37.6	11444.10	64.2	1.70463	748	22.124	66
568	22810.7	37.7	11508.30	64.7	1.71211	753	22.190	67
567	22848.4	37.7	11573.00	65.0	1.71964	757	22.257	66
566	22886.1	37.9	11638.00	65.5	1.72721	761	22.323	67
565	22924.0	37.9	11703.50	65.8	1.73482	764	22.390	67
564	22961.9	37.9	11769.3	66.3	1.74246	769	22.457	68
563	22999.8	38.0	11835.6	66.7	1.75015	773	22.525	67
562	23037.8	38.1	11902.3	67.1	1.75788	777	22.592	68
561	23075.9	38.2	11969.4	67.5	1.76565	781	22.660	68
560	23114.1	38.2	12036.9	67.9	1.77346	785	22.728	68
559	23152.3	38.3	12104.8	68.4	1.78131	790	22.796	69
558	23190.6	38.3	12173.2	68.8	1.78921	794	22.865	69
557	23228.9	38.4	12242.0	69.2	1.79715	798	22.934	69
556	23267.3	38.5	12311.2	69.7	1.80513	802	23.003	69

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
555	23305.8	38.6	12380.9	70.1	1.81315	807	23.073	69
554	23344.4	38.7	12451.0	70.5	1.82122	811	23.141	70
553	23383.1	38.7	12521.5	70.9	1.82933	816	23.211	70
552	23421.8	38.7	12592.4	71.4	1.83749	820	23.281	70
551	23460.5	38.9	12663.8	71.9	1.84569	824	23.351	71
550	23499.4	38.9	12735.7	72.3	1.85393	829	23.422	71
549	23538.3	39.0	12808.0	72.8	1.86222	833	23.493	71
548	23577.3	39.1	12880.8	73.2	1.87055	838	23.564	71
547	23616.4	39.1	12954.0	73.7	1.87893	843	23.635	72
546	23655.5	39.2	13027.7	74.1	1.88736	847	23.707	72
545	23694.7	39.3	13101.8	74.6	1.89583	852	23.779	72
544	23734.0	39.3	13176.4	75.1	1.90435	857	23.851	72
543	23773.3	39.4	13251.5	75.6	1.91292	861	23.923	73
542	23812.7	39.5	13327.1	76.0	1.92153	867	23.996	73
541	23852.2	39.6	13403.1	76.6	1.93020	871	24.069	73
540	23891.8	39.6	13479.7	77.0	1.93891	878	24.142	73
539	23931.4	39.7	13556.7	77.6	1.94767	881	24.215	74
538	23971.1	39.8	13634.3	78.0	1.95648	886	24.289	74
537	24010.9	39.9	13712.3	78.5	1.96534	891	24.363	74
536	24050.8	39.9	13790.8	79.0	1.97425	896	24.437	75
535	24090.7	40.0	13869.8	79.5	1.98321	901	24.512	75
534	24130.7	40.1	13949.3	80.1	1.99222	905	24.587	75
533	24170.8	40.1	14029.4	80.5	2.00127	912	24.662	75
532	24210.9	40.3	14109.9	81.1	2.01039	916	24.737	76
531	24251.2	40.3	14191.0	81.6	2.01955	921	24.813	76
530	24291.5	40.4	14272.6	82.1	2.02876	926	24.889	76
529	24331.9	40.5	14354.7	82.7	2.03802	932	24.965	77
528	24372.4	40.5	14437.4	83.2	2.04734	937	25.042	77
527	24412.9	40.6	14520.6	83.7	2.05671	942	25.119	77
526	24453.5	40.7	14604.3	84.3	2.06613	948	25.196	77
525	24494.2	40.8	14688.6	84.9	2.07561	954	25.273	78
524	24535.0	40.9	14773.5	85.4	2.08515	959	25.351	78
523	24575.9	40.9	14858.9	85.9	2.09474	964	25.429	78
522	24616.8	41.0	14944.8	86.5	2.10438	970	25.507	79
521	24657.8	41.1	15031.3	87.0	2.11408	975	25.586	79
520	24698.9	41.1	15118.3	87.6	2.12383	981	25.665	79
519	24740.0	41.2	15205.9	88.2	2.13364	987	25.744	80
518	24781.2	41.3	15294.1	88.8	2.14351	992	25.824	80
517	24822.5	41.4	15382.9	89.4	2.15343	999	25.904	80
516	24863.9	41.5	15472.3	90.0	2.16342	1004	25.984	81
515	24905.4	41.5	15562.3	90.5	2.17346	1010	26.065	81
514	24946.9	41.7	15652.8	91.2	2.18356	1015	26.146	81
513	24988.6	41.7	15744.0	91.7	2.19371	1022	26.227	81
512	25030.3	41.8	15835.7	92.3	2.20393	1028	26.308	82
511	25072.1	41.9	15928.0	93.0	2.21421	1034	26.390	82

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TABLE I.—Continued.

Z	S_z	Δ	A_z	Δ	I_z	Δ	T_z	Δ
510	25114.0	42.0	16021.0	93.6	2.22455	1040	26.472	82
509	25156.0	42.0	16114.6	94.2	2.23495	1046	26.554	83
508	25198.0	42.2	16208.8	94.8	2.24541	1052	26.637	83
507	25240.2	42.2	16303.6	95.5	2.25593	1059	26.720	83
506	25282.4	42.3	16399.1	96.1	2.26652	1065	26.803	84
505	25324.7	42.4	16495.2	96.8	2.27717	1071	26.887	84
504	25367.1	42.5	16592.0	97.4	2.28788	1077	26.971	84
503	25409.6	42.6	16689.4	98.0	2.29865	1084	27.055	85
502	25452.2	42.7	16787.4	98.7	2.30949	1091	27.140	85
501	25494.9	42.7	16886.1	99.4	2.32040	1097	27.225	85
500	25537.6	42.8	16985.5	100.1	2.33137	1104	27.310	86
499	25580.4	42.9	17085.6	100.7	2.34241	1110	27.396	86

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TABLE II.

Values of δ, the ratio of the density of half saturated air for a given temperature and barometric height to the density of half-saturated air for 15° C. (59° F.), and 750 mm. (29.5275 inches) barometric height. Computed by the formula $\delta = \frac{1.05498}{29.4338} \frac{H - \frac{3}{8} F_t}{.92485 + .002036\, t}$, in which H is barometric height in inches, t is temperature Fahrenheit, and F_t is the vapor pressure in saturated air at t°.

t_f	28 in.	29 in.	30 in.	31 in.	t_f	28 in.	29 in.	30 in.	31 in.
0	1.073	1.112	1.150	1.188	50	.966	1.000	1.035	1.069
1	1.071	1.110	1.148	1.186	51	.964	.998	1.033	1.067
2	1.069	1.108	1.146	1.184	52	.962	.996	1.031	1.065
3	1.066	1.105	1.143	1.181	53	.960	.994	1.029	1.063
4	1.064	1.103	1.140	1.178	54	.958	.992	1.027	1.061
5	1.062	1.100	1.137	1.175	55	.956	.990	1.024	1.058
6	1.060	1.098	1.135	1.173	56	.954	.988	1.022	1.056
7	1.057	1.095	1.132	1.170	57	.952	.986	1.020	1.054
8	1.055	1.093	1.130	1.168	58	.950	.984	1.018	1.052
9	1.052	1.090	1.127	1.165	59	.948	.982	1.016	1.050
10	1.050	1.088	1.125	1.163	60	.946	.980	1.014	1.048
11	1.048	1.086	1.123	1.161	61	.944	.978	1.012	1.046
12	1.046	1.084	1.121	1.159	62	.942	.976	1.010	1.044
13	1.043	1.081	1.118	1.156	63	.941	.974	1.008	1.042
14	1.041	1.079	1.116	1.153	64	.939	.972	1.006	1.040
15	1.039	1.077	1.113	1.150	65	.937	.970	1.003	1.037
16	1.037	1.074	1.110	1.147	66	.935	.968	1.001	1.035
17	1.035	1.072	1.108	1.145	67	.933	.966	.999	1.033
18	1.032	1.069	1.105	1.142	68	.931	.964	.997	1.031
19	1.030	1.067	1.103	1.140	69	.929	.962	.995	1.029
20	1.028	1.065	1.101	1.138	70	.927	.960	.993	1.027
21	1.026	1.063	1.099	1.136	71	.925	.958	.991	1.025
22	1.024	1.061	1.097	1.134	72	.923	.956	.989	1.023
23	1.021	1.058	1.094	1.131	73	.921	.954	.987	1.021
24	1.019	1.056	1.091	1.128	74	.919	.952	.985	1.019
25	1.017	1.053	1.088	1.125	75	.917	.950	.982	1.016
26	1.015	1.051	1.086	1.123	76	.915	.948	.980	1.014
27	1.013	1.049	1.084	1.121	77	.913	.946	.978	1.012
28	1.011	1.047	1.082	1.119	78	.912	.945	.977	1.010
29	1.009	1.045	1.080	1.117	79	.910	.943	.975	1.008
30	1.007	1.043	1.078	1.115	80	.908	.941	.973	1.006
31	1.005	1.041	1.076	1.113	81	.906	.939	.971	1.004
32	1.003	1.039	1.074	1.111	82	.904	.937	.969	1.002
33	1.000	1.036	1.071	1.108	83	.903	.935	.967	1.000
34	.998	1.034	1.069	1.105	84	.901	.933	.965	.998
35	.996	1.031	1.066	1.102	85	.899	.931	.963	.995
36	.994	1.029	1.064	1.100	86	.897	.929	.961	.993
37	.992	1.027	1.062	1.098	87	.895	.927	.959	.991
38	.990	1.025	1.060	1.096	88	.893	.925	.957	.989
39	.988	1.023	1.058	1.094	89	.891	.923	.955	.987
40	.986	1.021	1.056	1.092	90	.889	.921	.953	.985
41	.984	1.019	1.054	1.090	91	.887	.919	.951	.983
42	.982	1.017	1.052	1.088	92	.885	.917	.949	.981
43	.980	1.015	1.050	1.085	93	.884	.916	.947	.979
44	.978	1.013	1.048	1.083	94	.882	.914	.945	.977
45	.976	1.011	1.046	1.081	95	.880	.912	.943	.975
46	.974	1.008	1.043	1.078	96	.878	.910	.941	.973
47	.972	1.006	1.041	1.076	97	.876	.908	.939	.971
48	.970	1.004	1.039	1.073	98	.874	.906	.937	.969
49	.968	1.002	1.037	1.071	99	.872	.904	.935	.967
50	.966	1.000	1.035	1.069	100	.870	.902	.933	.965

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TABLE III.

Values of the *altitude factor* f, the ratio of the density of the air at the level of the gun to its density at the height h above the gun. Computed by the formula $f = \left(1 - .29\,\frac{h}{H}\right)^{-2.451}$, in which h is the height in feet and H = 27,700 is the height of a homogeneous atmosphere of the density of the air at the gun's level.

h	0	100	200	300	400	500	600	700	800	900
0	1.000	1.003	1.005	1.008	1.010	1.013	1.016	1.018	1.021	1.023
1000	1.026	1.029	1.031	1.034	1.037	1.039	1.042	1.045	1.048	1.050
2000	1.053	1.056	1.059	1.061	1.064	1.067	1.070	1.073	1.075	1.078
3000	1.081	1.084	1.087	1.090	1.093	1.096	1.099	1.102	1.105	1.108
4000	1.111	1.114	1.117	1.120	1.123	1.126	1.129	1.132	1.135	1.138
5000	1.141	1.144	1.147	1.150	1.153	1.156	1.160	1.163	1.166	1.169
6000	1.172	1.175	1.179	1.182	1.185	1.188	1.192	1.195	1.198	1.202
7000	1.205	1.208	1.212	1.216	1.219	1.222	1.225	1.229	1.232	1.236
8000	1.239	1.243	1.246	1.250	1.253	1.257	1.261	1.264	1.268	1.271
9000	1.275	1.279	1.282	1.286	1.290	1.293	1.297	1.301	1.305	1.308
10000	1.312	1.316	1.320	1.323	1.327	1.331	1.335	1.339	1.342	1.346
11000	1.350	1.354	1.358	1.362	1.366	1.370	1.374	1.378	1.382	1.386
12000	1.390	1.394	1.398	1.402	1.406	1.410	1.415	1.419	1.423	1.427

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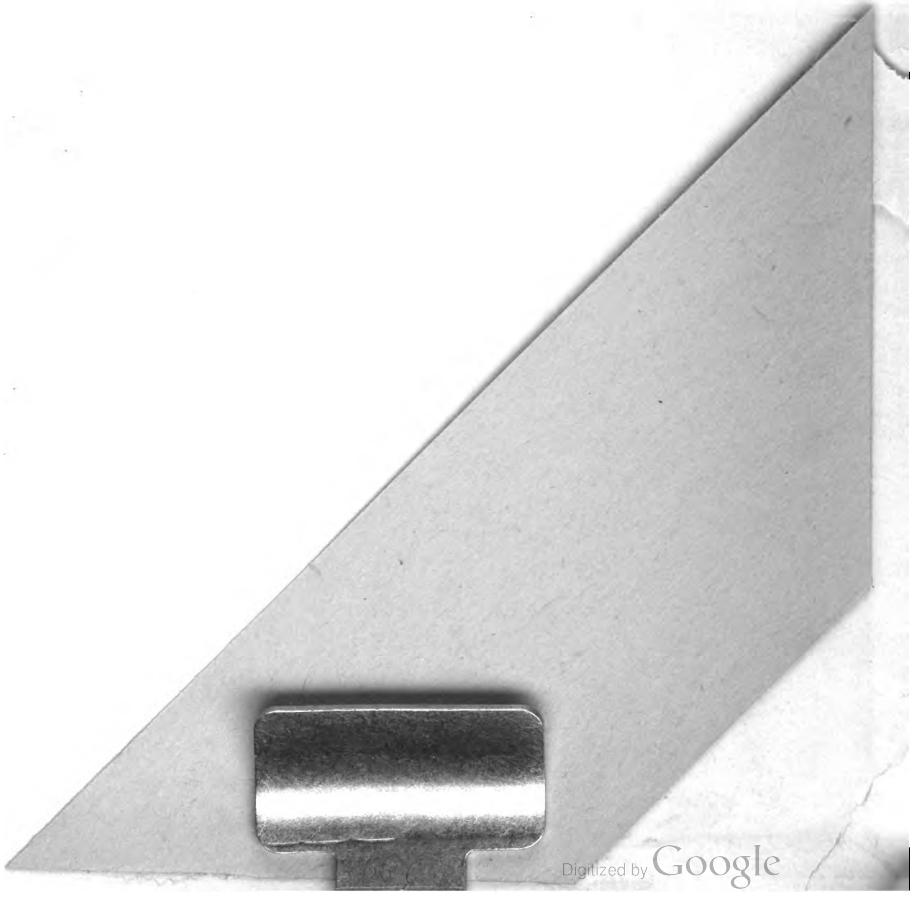

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